Lecture 2: The Financial Econometrics of Option Markets

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Day 1:

1. Introduction to options
2. Basic pricing ideas
3. Econometric interpretation to pricing
4. Specification of price dynamics
5. The Black-Scholes model
6. Eyeballing options data
7. Estimation
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Day 2:

1. Testing the Black-Scholes model
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   (c) Smiles and smirks

2. Relaxing the assumptions of Black-Scholes
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   (b) Pricing skewness and kurtosis
   (c) Functional forms and nonparametrics
   (d) Time-varying volatility
       (GARCH, stochastic volatility, forward looking)
   (e) Jumps
       (Poisson, Hawkes)

3. Monte Carlo pricing methods

4. Application to Exchange options and contagion
Testing the Black-Scholes Model

- Consider the empirical model based on Black-Scholes
  \[ c_i = c_i^{BS}(\sigma) + u_i \]
  where \( c_i \) represents the observed price of the \( i^{th} \) option and \( c_i^{BS}(\sigma) \) is the price of the \( i^{th} \) option contract based on Black-Scholes
  \[ c_i = p_i \Phi(d_i) - k_i e^{-r_i} \Phi(d_i - \sigma \sqrt{h_i}) \]
  with \( d_i \) defined as
  \[ d_i = \frac{\log(p_i / k_i) + (r_i + \frac{1}{2} \sigma^2) h_i}{\sigma \sqrt{h_i}} \]

- The Black-Scholes model imposes very strong restrictions on the underlying process which can be tested using various inferential methods.

- Three tests of the model are now investigated.
  1. Biasedness
  2. Heteroskedasticity
  3. Smiles and Smirks
The empirical model does not contain an intercept in the conditional mean suggesting that the conditional mean of the call option price equals the Black-Scholes price

\[ E[c_i | p_i] = c_i^{BS} = p_i \Phi(d_i) - k_i e^{-r_{hi}} \Phi(d_i - \sigma \sqrt{h_i}) \]

as \( E[u_i] = 0 \).

This result is interpreted as the call option price is unbiased. To allow for the possibility of the observed option price being biased the empirical model is extended to include an intercept

\[ c_i = \beta_0 + p_i \Phi(d_i) - k_i e^{-r_{hi}} \Phi(d_i - \sigma \sqrt{h_i}) + u_i \]

where \( \beta_0 \) is the unknown intercept. A test of the unbiasedness property is that the intercept is zero is based on the hypothesis

\[ \beta_0 = 0 \]

This restriction can be tested using a standard t-test.
Example (Testing for Biasedness)

The number of iterations for the algorithm to converge is 142 yielding a log-likelihood value of

\[
\log L \left( \hat{\theta} \right) = -31050.49
\]

The parameter estimates are

\[
\hat{\theta} = \left\{ \hat{\sigma}^2 = 0.014184, \quad \hat{\omega}^2 = 0.551221, \quad \hat{\beta}_0 = 0.810496 \right\}
\]

As \( se(\hat{\beta}_0) = 0.009125 \), a test of biasedness is given by the t-statistic

\[
tstat = \frac{0.810496 - 0.0}{0.009125} = 88.818
\]

This constitutes significant bias and a rejection of Black-Scholes.
Nonetheless, the volatility estimate is similar to when $\beta_0 = 0$ as the estimate of the annualised volatility is

$$\hat{\sigma} = \sqrt{0.014184} = 0.1191$$

or 11.91% compared to 13.05% with $\beta_0 = 0$. 
The biasedness property can be extended to allow for different biasedness for different contracts.

This is achieved by including a dummy variable for different strike prices. As there is an intercept and there are 30 strike prices the regression equation is augmented to include just the first 29 strike prices to circumvent the dummy variable trap.

Alternatively, an equivalent approach is to include all 30 strike price dummy variables and exclude the intercept.
The empirical model is now expanded as

$$c_i = \beta_0 + \sum_{j=1}^{29} \beta_j D_{j,i} + p_i \Phi(d_i) - k_i e^{-r_i h_i} \Phi(d_i - \sigma \sqrt{h_i}) + u_i$$

where $D_{j,i}, j = 1, 2, \cdots, 29$, are dummy variables defined as a dummy variable corresponding to the highest strike price of $k = 550$, is not needed as the equation contains an intercept term, thereby avoiding the

$$D_{1,i} = \begin{cases} 1 & : k = 350 \\ 0 & : \text{otherwise} \end{cases}$$

$$D_{2,i} = \begin{cases} 1 & : k = 370 \\ 0 & : \text{otherwise} \end{cases}$$

$$\vdots$$

$$D_{29,i} = \begin{cases} 1 & : k = 530 \\ 0 & : \text{otherwise} \end{cases}$$
Instead of constructing dummy variables for each strike price, dummy variables can be grouped together in terms of “moneyness”. A possible classification of moneyness is:

\[
DUM_{\text{OUT}}_j = \begin{cases} 
1 : & \frac{p}{k} < 0.97 \\
0 : & \text{otherwise}
\end{cases}
\]

\[
DUM_{\text{AT}}_j = \begin{cases} 
1 : & 0.97 < \frac{p}{k} < 1.03 \\
0 : & \text{otherwise}
\end{cases}
\]

\[
DUM_{\text{IN}}_j = \begin{cases} 
1 : & 1.03 < \frac{p}{k} \\
0 : & \text{otherwise}
\end{cases}
\]

If the model contains an intercept term then again it will only be necessary to include two of the moneyness variables.
Testing the Black-Scholes Model

Biasedness

- Instead of just including dummy variables based on strike prices, dummy variables corresponding to other characteristics of option contracts can be used to augment the model. For example, dummy variables on the three types of maturities in the stock option data set would be defined as

\[
D_{MAY,i} = \begin{cases} 
1 & : h_i = 0.123288 \\
0 & : \text{otherwise} 
\end{cases}
\]

\[
D_{JUNE,i} = \begin{cases} 
1 & : h_i = 0.200000 \\
0 & : \text{otherwise} 
\end{cases}
\]

\[
D_{SEPT,i} = \begin{cases} 
1 & : h_i = 0.457534 \\
0 & : \text{otherwise} 
\end{cases}
\]

(1)

corresponding respectively to May, June and September option contracts.

- As before, only two of the three dummy variables need to be included if the model already contains an intercept term.
An important assumption of the empirical model is that the variance of the disturbance term $\omega^2$, is constant across all contract types.

To relax the assumption of homoskedasticity the dummy variables defined for strikes and maturities in can be used to allow the disturbance variance to change over the sample. In the case of the maturity dummy variables, the pricing error disturbance term is specified as

$$\omega_i^2 = \exp (\alpha_0 + \alpha_1 D_{MAY,i} + \alpha_2 D_{JUNE,i})$$

A joint test of the restrictions

$$\alpha_1 = \alpha_2 = 0$$

is a test of homoskedasticity which can be performed using a likelihood ratio test for example.
In estimating the option price models, the data consist of option contracts corresponding to 30 strike prices and 3 maturities, a total of 90 different types of call option contracts.

An important feature of the estimated model is that there is just a single estimate of the volatility parameter $\sigma$, regardless of the type of option contract.

This is a restriction of the model which can be tested by reestimating the model for alternative strike prices and alternative maturities.
A test of the assumption that \( \sigma \) is invariant to strike prices \( k \) is to group the data for each \( k \) and reestimate \( \sigma \) for each group.

1. If the Black-Scholes model is consistent with the data then the estimates of \( \sigma \) for each group should not be statistically different from each other.
2. If the estimates \( \sigma \) are different across groups this is evidence against the Black-Scholes model.

In practice there are two types of relationships between \( \sigma \) and \( k \).

1. Volatility Smile - there is a U-shape relationship between \( \sigma \) and \( k \), centered around the at-the-money options. Occurs in currency options.
2. Volatility Smirk or skew - there is an inverse relationship between \( \sigma \) and \( k \), with volatility being relatively high for in-the-money options and low for out-of-the-money options. Occurs in stock options.
Example (Volatility Smirk in Stock Options)

The model is estimated using option prices separately for each of the 30 groups of strike prices for the European call options written on the S&P500 stock index on April 4, 1995. The fitted line is based on a nonparametric kernel regression. There is evidence of a smirk with estimates of $\sigma$ ranging from 0.3 to 0.1. This result is evidence of misspecification.
Evidence of volatility smiles and smirks in particular have led to the specification of more general models of asset prices.

Some potential specifications are:

1. Relaxing the assumption of lognormality and adopting a more general specification.
2. Relaxing the assumption of constant volatility.
3. Allow for additional effects in the mean.
4. Allow for additional effects in the variance of the pricing error (heteroskedasticity).
Melick and Thomas (1997) specify a mixture of lognormal distributions.

The empirical option pricing model is

\[ c_i = \alpha c_{1,i}^{BS}(\sigma_1) + (1 - \alpha) c_{1,i}^{BS}(\sigma_2), \]

where \( BS(\sigma_j), j = 1, 2, \) is the Black-Scholes price

\[ c_{j,i}^{BS} = p_i \Phi(d_{j,i}) - k_i e^{-r_i h_i} \Phi(d_{j,i} - \sigma_j \sqrt{h_i}) \]

with constant volatility, \( \sigma_j, j = 1, 2 \) and

\[ d_{j,i} = \frac{\log(p_i / k_i) + (r_i + \frac{1}{2} \sigma_j^2) h_i}{\sigma_j \sqrt{h_i}} \]

The parameter \( 0 \leq \alpha \leq 1, \) is the mixing parameter which weights the two subordinate lognormal distributions.
Example (Estimating the Mixture Model)

The number of iterations for the algorithm to converge is 76 yielding a log-likelihood value of

\[ \log L (\hat{\theta}) = -12142.95 \]

The parameter estimates are

\[ \hat{\theta} = \{ \hat{\sigma}_1^2 = 0.000917, \hat{\sigma}_1^2 = 0.042563, \hat{\alpha} = 0.551565, \hat{\omega}^2 = 0.140724 \} \]

The estimates of the volatilities of the two distributions, \( \sigma_1^2 \) and \( \sigma_1^2 \), suggest that the two lognormal distributions have very different shapes.
Example (Estimating the Mixture Model)

The estimate of \( \alpha \) is 0.551565 suggesting that these two distributions have roughly equal weights. A test of equal weights given by the t-statistic

\[
tstat = \frac{0.551565 - 0.5}{0.001567} = 32.907
\]

suggests a rejection of the hypothesis \( \alpha = 0.5 \) at conventional significance levels.
The class of semi-nonparametric option pricing models discussed here are based on an augmentation of the normal returns density through the inclusion of higher order terms.

Jarrow and Rudd (1982) were the first to adopt this approach, which was implemented by Corrado and Su (1997) and Capelle-Blancard, Jurczenko and Maillet (2001).

Let $g(z_{t+h})$ represent the true conditional price distribution of the standardised returns

$$z_{t+h} = \frac{\log z_{t+h} - \log z_t - (r - \frac{1}{2} \sigma^2) h}{\sigma \sqrt{h}}$$
Using an Edgeworth expansion of $g(z_{t+h})$ around the normal density gives

$$
g(z_{t+h}) = \phi(z_{t+h}) - \frac{\kappa_3(G) - \kappa_3(\Phi)}{3!} \frac{d^3\phi(z_{t+h})}{dz^3_{t+h}}$$

$$+ \frac{\kappa_4(G) - \kappa_4(\Phi)}{4!} \frac{d^4\phi(z_{t+h})}{dz^4_{t+h}} + \varepsilon(z_{t+h})$$

where $\varepsilon(z_{t+h})$ is an approximation error arising from the exclusion of higher order terms in the expansion and $\kappa_i$ is the $i^{th}$ cumulant of the associated distribution with $\kappa_1 = 0$ and $\kappa_2 = 1$ to standardize the distribution to have zero mean and unit variance.
Using the properties of the normal distribution

\[
g (z_{t+h}) = \phi (z_{t+h}) \left[ 1 + \gamma_1 \frac{(z_{t+h}^3 - 3z_{t+h})}{6} \\
+ \gamma_2 \frac{(z_{t+h}^4 - 6z_{t+h}^2 + 3)}{24} \right] + \varepsilon (z_{t+h})
\]

where

\[
\gamma_1 = \frac{(\kappa_3 (G) - \kappa_3 (\Phi))}{3!}, \quad \gamma_2 = \frac{(\kappa_4 (G) - \kappa_4 (\Phi))}{4!}
\]

are the unknown parameters which capture respectively skewness and kurtosis.

The expression shows that the difference between the true density, \( g (z_{t+h}) \) and the normal density \( \phi (z_{t+h}) \), is determined by the third and fourth moments.
As from the transformation of variable technique

\[ f(p_{t+h}|p_t) = |J| \phi(z_{t+h}) \]

where \( J \) is the Jacobian of the transformation from \( z_{t+h} \) to \( p_{t+h} \), then

\[ f(p_{t+h}|p_t) = |J| \phi(z_{t+h}) \left[ 1 + \gamma_1 \frac{(z_{t+h}^3 - 3z_{t+h})}{6} \right. \\
\left. + \gamma_2 \frac{(z_{t+h}^4 - 6z_{t+h}^2 + 3)}{24} \right] \]

where for simplicity the approximation error is ignored.
Using this density to price options yields the Jarrow-Rudd option pricing model:

\[ c_i^{JR} = c_i^{BS} + \gamma_1 Q_{3,i} + \gamma_2 Q_{4,i} \]

where \( c_i^{BS} \) is the Black-Scholes price for the \( i^{th} \) contract and

\[ Q_{3,i} = e^{-rh} \int_k^\infty (p_{t+h} - k) |J| \frac{(z_{t+h}^3 - 3z_{t+h})}{6} \phi dp_{t+h} \]

and

\[ Q_{4,i} = e^{-rh} \int_k^\infty (p_{t+h} - k) |J| \frac{(z_{t+h}^4 - 6z_{t+h}^2 + 3)}{24} \phi dp_{t+h} \]
Potential problem: the risk neutral probability distribution in is not constrained to be positive over the support of the density.

Interestingly, this problem appears to have been ignored in the literature so far. The problem arises as the returns distribution is a function of cubic and quartic polynomials, which can, in general, yield negative values.

The problem of negativity can be expected to be more severe when the Jarrow-Rudd approximation does not model the true distribution accurately, causing the polynomial terms to over-adjust, especially in the tails of the estimated distribution.
To impose non-negativity on the underlying risk neutral probability distribution the semi-nonparametric density of Gallant and Tauchen (1989) is specified

\[ p(z_{t+h}) = \Phi(z_{t+h}) \left[ 1 + \lambda_1 \frac{(z_{t+h}^3 - 3z_{t+h})}{6} + \lambda_2 \frac{(z_{t+h}^4 - 6z_{t+h}^2 + 3)}{24} \right]^2 \]

where the augmenting polynomial is now squared, forcing the probabilities to be greater than or equal to zero for all values of \( z_{t+h} \).

Using this expression for the standardised returns yields the semi-nonparametric option pricing model which is referred to as the SNP option pricing model.
Relaxing the Black-Scholes Assumptions
Nonparametric Pricing Based on Artificial Neural Networks

- The Black-Scholes option price shows that there is a nonlinear relationship between the option price $c^_{BS}_t$, and the remaining arguments, $p_t, k, h, r, \sigma$.
- For more general option price models it is not always possible to derive an analytic expression for the price of the option. One way to proceed is to use nonparametric methods based on for example, artificial neural networks (ANN).
- An example of an ANN model of option prices is given by

$$c_i = \alpha_0 + \alpha_1 (p_i - k_i) + \alpha_2 h_i + \alpha_3 L_i + v_i$$  \hspace{1cm} (2)

where

$$L_i = \frac{1}{1 + \exp \left[ - (\gamma_0 + \gamma_1 (p_i - k_i) + \gamma_2 h_i) \right]}$$

and $v_i$ is a disturbance term. The function $L_i$ represents the artificial neural network where the form of the ANN is the logistic squasher.
To estimate the parameters of the model in (2) an iterative maximum likelihood estimator can be used. Equivalently, the parameters can be chosen using MLE or even more simply a standard nonlinear least squares algorithm to minimise

$$\sum_{i=1}^{N} v_i^2$$

where $N$ is the number of option contracts in the data set.

The option prices are given by the fitted values

$$\hat{c}_i = \hat{\alpha}_0 + \hat{\alpha}_1 (p_i - k_i) + \hat{\alpha}_2 h_i + \hat{\alpha}_3 \hat{L}_i$$

where

$$\hat{L}_i = \frac{1}{1 + \exp \left[- (\hat{\gamma}_0 + \hat{\gamma}_1 (p_i - k_i) + \hat{\gamma}_2 h_i)\right]}$$
Having estimated the ANN it is possible to compute various hedge parameters to be used in a risk management strategy.

An estimate of the options’s delta is given by

\[
\frac{\partial \hat{c}_i}{\partial p} = \hat{\alpha}_1 + \hat{\alpha}_3 \hat{\gamma}_1 \hat{l}_i
\]

where

\[
\hat{l}_i = \frac{\exp \left[ - (\hat{\gamma}_0 + \hat{\gamma}_1 (p_i - k_i) + \hat{\gamma}_2 h_i) \right]}{1 + \exp \left[ - (\hat{\gamma}_0 + \hat{\gamma}_1 (p_i - k_i) + \hat{\gamma}_2 h_i) \right]^2}
\]

which is the logistic density function.
Relaxing the Black-Scholes Assumptions
Time-varying Volatility

GARCH (Engle and Mustafa)

- A common method to allow for time-varying volatility is to use a GARCH specification.
- The model of asset prices over a day ($\Delta = 1/250$) is

$$
\log p_{t+1} - \log p_t = \left( r - \frac{1}{2} \sigma_t^2 \right) \Delta + \sqrt{\sigma_t^2 \Delta} z_{t+1}
$$

$$
\sigma_{t+1}^2 = \alpha_0 + \alpha_1 \sigma_t^2 z_t^2 + \beta_1 \sigma_t^2
$$

$$
z_{t+1} \sim \mathcal{N}(0, 1)
$$

- The variance is time-varying due to
  1. Lagged squared shocks from the mean $\alpha_1 \sigma_t^2 z_t^2$. Allows for memory in the variance to be one period.
  2. Lagged conditional variance $\beta_1 \sigma_t^2$. Allows for memory in the variance to be longer than one period.

- But pricing options is more difficult as there is no closed form solution for the option price. Can use Monte Carlo methods.
GARCH (Heston and Nandi)

- An alternative GARCH model to Engle and Mustafa that yields an analytical formula for pricing European options.
- The model of asset prices over a day ($\Delta = 1/250$) is

$$
\log p_{t+1} - \log p_t = (r + \gamma \sigma_t^2) \Delta + \sqrt{\sigma_t^2 \Delta} z_{t+1} \\
\sigma_{t+1}^2 = \beta_0 + \beta_1 \sigma_t^2 + \beta_2 (z_t - \beta_3 \sigma t)^2 \\
z_{t+1} \sim N(0, 1)
$$

- Under risk neutrality the model becomes

$$
\log p_{t+1} - \log p_t = r - \frac{1}{2} \sigma_t^2 + \sqrt{\sigma_t^2 \Delta} z^*_{t+1} \\
\sigma_{t+1}^2 = \beta_0 + \beta_1 \sigma_t^2 + \beta_2 (z^*_t - \beta_3 \sigma_t)^2 \\
z^*_{t+1} \sim N(0, 1)
$$

where the $z^*_{t+1} = z_{t+1} + (\lambda + 1/2) \sigma_t$ under the risk neutral measure

- A quasi closed form expression is now available to price options.
**Stochastic Volatility (Heston)**

- A common method to allow for time-varying volatility is the stochastic volatility model.
- The model of asset prices over a day ($\Delta = 1/250$) is

$$
\log p_{t+1} - \log p_t = \left( r - \frac{1}{2} \sigma^2_t \right) \Delta + \sqrt{\sigma^2_t} \Delta z_{1,t+1}
$$

$$
\sigma^2_{t+1} - \sigma^2_t = \kappa \left( \phi - \sigma^2_t \right) \Delta + \sqrt{\beta} \Delta \sigma^2_t z_{2,t+1}
$$

$$
\begin{bmatrix}
z_{1,t+1} \\
z_{2,t+1}
\end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)
$$

- This model introduces an additional disturbance term $z_{2,t}$, in contrast to GARCH volatility models. It is for this reason why stochastic volatility models tend to be more difficult to estimate.
Stochastic Volatility (Heston)

- The volatility process is mean reverting to the long-run parameter $\phi$, with the rate of conversion controlled by $\kappa$.
- The strength of the stochastic volatility is controlled by $\beta$, i.e., the parameter that controls the volatility of volatility. The volatility also displays the CIR specification with $\sqrt{\beta \Delta \sigma_t^2}$. (Model specification used by Heston (1993)).
- The two stochastic processes are allowed to be correlated with parameter $\rho$. Sometimes referred to as the “leverage” effect.
- Heston derives European option prices for this model, although numerical procedures are needed to evaluate the option prices.
Forward Looking Volatility (Engle and Rosenberg)

- The GARCH volatility specification is backward looking as $\sigma_t^2$ is a function of past shocks.
- A specification that is a function of future shocks is
  \[
  \sigma_{t+h|t} = \exp(\beta_1 + \beta_2 \ln(p_{t+h}/p_t))
  \]
- This specification shows that conditional volatility is stochastic, as it is a function of the future return over the life of the option, $\ln(p_{t+h}/p_t)$.
- Special cases
  1. $\beta_2 > 0$: the relationship between volatility and future return is positive.
  2. $\beta_2 < 0$: there is an inverse relationship between future return and volatility (leverage effect).
  3. $\beta_2 = 0$: constant volatility specification which underlies the Black-Scholes model.
Forward Looking Volatility (Engle and Rosenberg)

- A more natural specification to adopt in the context of option pricing as option prices are based on an evaluation of the future evolution of the underlying spot price.

- In common with a GARCH-type volatility specification, an additional error term is not introduced, with all randomness deriving from randomness in the asset price itself. This contrasts with a stochastic volatility model, in which the volatility process has its own random innovations.

- This specification has computational advantages compared with the GARCH and stochastic volatility specifications.
Poisson Processes

- The aim is to allow for infrequent jumps in asset prices.
- The first form of jumps is based on a Poisson process. The model of asset prices over a day ($\Delta = 1/250$) is

\[
\log p_{t+1} - \log p_t = \left( r - \frac{1}{2} \sigma^2 \right) \Delta + \sqrt{\sigma^2} \Delta z_{t+1} + J \Delta N_{t+1}
\]

\[
z_{t+1} \sim N(0, 1)
\]

\[
dN_{t+1} \sim Po(\lambda)
\]

where the size of the jump is $J$ and $\Delta N_{t+1}$ is a Poisson process with jump intensity parameter $\lambda$ given by

\[
P(\Delta N_{t+1} = 0) = 1 - \lambda \Delta + o(\Delta)
\]

\[
P(\Delta N_{t+1} = 1) = \lambda \Delta + o(\Delta)
\]

\[
P(\Delta N_{t+1} > 1) = o(dt).
\]

The larger is $\lambda$, the more likely a jump will occur.
Poisson Processes

- Note that the probabilities are a function of the time interval $\Delta$ which in general will be small. This means that:
  
  1. The occurrence of no jump is relatively “high” with probability of $1 - \lambda \Delta$.
  2. The occurrence of one jump is relatively “low” with probability of $\lambda \Delta$.
  3. The occurrence of more than one jump is effectively zero.

- A problem with the Poisson process is that the jumps are independent. This is especially not the case during financial crises where there tends to be a sequence of related jumps.
Hawkes Processes

- One way to relax the assumption that jumps are independent is to use a Hawkes process.
- The model of asset prices over a day ($\Delta = 1/250$) is

$$
\log p_{t+1} - \log p_t = \left( r - \frac{1}{2} \sigma^2 \right) \Delta + \sqrt{\sigma^2} \Delta z_{t+1} + J \Delta N_{t+1}
$$

$$
\lambda_{t+1} - \lambda_t = \alpha_1 (\lambda - \lambda_t) \Delta + \beta \Delta N_{t+1}
$$

$$
z_{t+1} \sim N(0, 1)
$$

$$
\Delta N_{t+1} \sim Po(\lambda_t)
$$

- The jump intensity parameter is now time-varying as given by the second equation.
Hawkes Processes

- An example of a simulated time series with jumps.
Monte Carlo Pricing Methods

- A feature of the option price solutions is that as the assumptions of the Black-Scholes model are relaxed, such as log-normality, constant volatility and no jumps, the calculation of the option price quickly becomes more difficult using analytical methods, if not impossible.
- In the latter situation numerical methods based on Monte Carlo simulations are available.
- Consider the 5 simulated time paths of the asset reported in the previous lecture, but reported here again for convenience.
Monte Carlo Pricing Methods

- This Figure 40 was based on simulating the model

\[
\log p_{t+h} - \log p_t = \left( r - \frac{1}{2} \sigma^2 \right) \Delta + \sqrt{\sigma^2 h} z_{t+h}
\]

\[
z_{t+h} \sim N(0,1)
\]

where the interest rate is \( r = 0.05 \), the volatility is \( \sigma = 0.2 \) and \( \Delta = 1/250 \) represents daily movements.

- For a 6-month option (\( h = 0.5 \)) with a strike price of \( k = 4.8 \), the Black-Scholes call option price was computed numerically as

\[
c_t = \exp(-rh) \frac{1}{N} \sum_{i=1}^{N} \max(p_{t+h} - k, 0) = 0.5025
\]

with \( N = 5 \).
The effect of increasing the number of simulation paths \( N \), on the price of the option are summarised in following Table.

<table>
<thead>
<tr>
<th>Number of Simulation Paths ( (N) )</th>
<th>Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5025</td>
</tr>
<tr>
<td>10</td>
<td>0.3822</td>
</tr>
<tr>
<td>50</td>
<td>0.4248</td>
</tr>
<tr>
<td>100</td>
<td>0.4757</td>
</tr>
<tr>
<td>200</td>
<td>0.4639</td>
</tr>
<tr>
<td>Analytical</td>
<td>0.4611</td>
</tr>
</tbody>
</table>
For comparison the Black-Scholes call option price based on the analytical solution is

\[
\begin{align*}
  c_t &= p_t \Phi(d) - ke^{-rh} \Phi(d - \sigma\sqrt{h}) \\
  &= 5.00 \times 0.7041 - 4.80 \times \exp(-0.05 \times 0.5) \times 0.6535 \\
  &= 0.4611
\end{align*}
\]

where

\[
\begin{align*}
  d &= \log(p_t/k) + (r + \frac{1}{2} \sigma^2)h \\
  &= \frac{\log(5.00/4.80) + (0.05 + 0.5 \times 0.2^2) \times 0.5}{0.2 \times \sqrt{0.5}} \\
  &= 0.5361
\end{align*}
\]

while \( \Phi(d) = \Phi(0.5361) = 0.7041 \) and 
\( \Phi(d - \sigma\sqrt{h}) = \Phi(0.5361 - 0.2\sqrt{0.5}) = \Phi(0.3947) = 0.6535 \).
The results show that as the number of simulation paths increases the numerical price becomes more accurate. When the number of sample paths increases to $N = 200$, the numerical price is $0.4639$, compared to the analytical price of $0.4611$, an error of less than 1%.

Monte Carlo methods can be made more accurate through

1. Antithetic variates (resimulate the model changing the sign of the residuals)
2. Control variates (compute the bias between an analytical solution and its simulated value).

Can price options where

1. Stochastic volatility
2. Jumping behaviour
3. Nonnormalities
Asset mispricing can be particularly significant during periods of financial crises and contagion.

Co-dependence structures across financial markets change dramatically during periods of financial turbulence that extends beyond the usual changes in volatilities and correlations.

Additional crisis transmission channel operating through higher order co-moments of asset returns (see Fry, Martin and Tang (JBES, 2010)), proved to be significant during many financial crises.

Important implications for market participants engaged in the hedging of financial risks and for financial regulators seeking to manage risks across the financial institutions.
### Descriptive Statistics

<table>
<thead>
<tr>
<th>Country</th>
<th>Noncrisis</th>
<th>Subprime</th>
<th>Great Recession</th>
<th>European Debt</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>US</strong></td>
<td>Mean</td>
<td>0.062</td>
<td>-0.063</td>
<td>-0.027</td>
</tr>
<tr>
<td></td>
<td>Std dev.</td>
<td>0.629</td>
<td>1.235</td>
<td>2.246</td>
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<td></td>
<td>Skewness</td>
<td>-0.545</td>
<td>-0.040</td>
<td>0.111</td>
</tr>
<tr>
<td><strong>France</strong></td>
<td>Mean</td>
<td>0.087</td>
<td>-0.089</td>
<td>-0.025</td>
</tr>
<tr>
<td></td>
<td>Std dev.</td>
<td>1.010</td>
<td>1.593</td>
<td>2.887</td>
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<tr>
<td></td>
<td>Skewness</td>
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<td>-0.200</td>
<td>0.177</td>
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<tr>
<td><strong>Greece</strong></td>
<td>Mean</td>
<td>0.117</td>
<td>-0.153</td>
<td>-0.099</td>
</tr>
<tr>
<td></td>
<td>Std dev.</td>
<td>1.196</td>
<td>1.707</td>
<td>2.896</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
<td>-0.479</td>
<td>0.049</td>
<td>-0.202</td>
</tr>
</tbody>
</table>
Statistics on coskewness

<table>
<thead>
<tr>
<th></th>
<th>Noncrisis</th>
<th>Subprime</th>
<th>Great Recession</th>
<th>European Debt</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>US (i = 1)</strong></td>
<td></td>
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<tr>
<td>France</td>
<td>-0.200</td>
<td>0.089</td>
<td>0.001</td>
<td>-0.030</td>
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<tr>
<td>Greece</td>
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<td>0.237</td>
<td>-0.238</td>
<td>0.095</td>
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<tr>
<td><strong>France (i = 1)</strong></td>
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<td></td>
</tr>
<tr>
<td>Greece</td>
<td>-0.203</td>
<td>-0.008</td>
<td>-0.016</td>
<td>0.120</td>
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<tr>
<td>US</td>
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<td>-0.140</td>
<td>-0.208</td>
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<tr>
<td><strong>Greece (i = 1)</strong></td>
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<tr>
<td>France</td>
<td>-0.168</td>
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<td>0.040</td>
<td>0.035</td>
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<tr>
<td>US</td>
<td>-0.296</td>
<td>-0.049</td>
<td>-0.217</td>
<td>-0.127</td>
</tr>
</tbody>
</table>
Objectives

- Investigate the implications of changes in the co-moments of the returns distribution that arise from contagion, in the context of pricing and hedging exchange options.
- Examine the potential size of mispricing from ignoring higher order co-moments of asset returns by comparing the simulated prices that allow for coskewness with the Black-Scholes price during periods of crises and contagion.
- Potential hedging losses from using the deltas that are derived based on the Black-Scholes price.
Application to Exchange Options and Contagion

Approach

- Use exchange options to capture co-dependence.
- Noncrisis period is given by Black-Scholes where returns $y_{i,t} = \log p_{i,t} - \log p_{i,t-1}$, are bivariate normal.
- Crisis period is represented by returns being distributed as bivariate generalized normal (GEN)

$$f(r_1, r_2) = \exp \left[ \theta_1 r_1^2 + \theta_2 r_2^2 + \theta_3 r_1 r_2 + \theta_4 r_1^3 + \theta_5 r_1^2 r_2 \\
+ \theta_6 r_1 r_2^2 + \theta_7 r_2^3 + \theta_8 r_1^4 + \theta_9 r_2^4 - \eta \right],$$

where $\eta$ is the normalising constant such that $\int \int f(r_1, r_2) \, dr_1 r_2 = 1$.

- To compute the option price assuming that returns are based on the generalized normal distribution, option prices are computed using Monte Carlo methods with $N = 10000$ simulation paths

$$C_t^{GEN} = \exp \left[ -rph \right] \frac{1}{N} \sum_{i=1}^{N} \max \left( p_{1,t+h}^i - p_{2,t+h}^i, 0 \right)$$
Associated with the bivariate GEN distribution of returns, there is the bivariate generalised lognormal distribution.

Using the transformation of variable technique

\[ g(p_{1,t}, p_{2,t}) = \begin{vmatrix} J \end{vmatrix} f(\ln p_{1,t} - \ln p_{1,t-1}, \ln p_{2,t} - \ln p_{2,t-1}) \]

where the Jacobian is

\[ \begin{vmatrix} J \end{vmatrix} = \begin{vmatrix} 1 \\ \frac{1}{p_{1,t} p_{2,t}} \end{vmatrix} \]
The bivariate distribution of prices conditional on lagged prices is

\[
g(p_{1,t}, p_{2,t} \mid p_{1,t-1}, p_{2,t-1}) = \exp \left[ \theta_1 (\log p_{1,t} - \log p_{1,t-1})^2 + \theta_2 (\log p_{2,t} - \log p_{2,t-1})^2 \\
+ \theta_3 (\log p_{1,t} - \log p_{1,t-1}) (\log p_{2,t} - \log p_{2,t-1}) \\
+ \theta_4 (\log p_{1,t} - \log p_{1,t-1})^3 \\
+ \theta_5 (\log p_{1,t} - \log p_{1,t-1})^2 (\log p_{2,t} - \log p_{2,t-1}) \\
+ \theta_6 (\log p_{1,t} - \log p_{1,t-1}) (\log p_{2,t} - \log p_{2,t-1})^2 \\
+ \theta_7 (\log p_{2,t} - \log p_{2,t-1})^3 + \theta_8 (\log p_{1,t} - \log p_{1,t-1})^4 \\
+ \theta_9 (\log p_{2,t} - \log p_{2,t-1})^4 - \eta \right] \frac{1}{p_{1,t}p_{2,t}}
\]
The following experiments are conducted.

1. Effect of mispricing of options when incorrectly using Black-Scholes when the true distribution is bivariate GEN.

2. Effect on delta hedging when assume the wrong distribution.

3. Effect on replicating risk free portfolios from incorrect choice of distribution.