Lecture 1: The Financial Econometrics of Option Markets

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Day 1:

1. Introduction to options
2. Basic pricing ideas
3. Econometric interpretation to pricing
4. Specification of price dynamics
5. The Black-Scholes model
6. Eyeballing options data
7. Estimation
   (a) Using returns data
   (b) Using options data
   (c) Using returns and options data
Day 2:

1. Testing the Black-Scholes model
   (a) Biasedness
   (b) Heteroskedasticity
   (c) Smiles and smirks

2. Relaxing the assumptions of Black-Scholes
   (a) Distribution
   (b) Pricing skewness and kurtosis
   (c) Functional forms and nonparametrics
   (d) Time-varying volatility
      (GARCH, stochastic volatility, forward looking)
   (e) Jumps
      (Poisson, Hawkes)

3. Monte Carlo pricing methods

4. Application to Exchange options and contagion
Options were first traded on April 26, 1973, at the Chicago Board Options Exchange (CBOE) - the number of traded contracts was 911. Since then the options market has grown enormously. The Figure shows the number of contracts traded daily on options written on the S&P500 index, increased from 87,286 in 2000 to 783,768 in 2011.

Reason: option contracts provide a more flexible way to manage risks than more conventional assets including stocks and bonds.
Introduction

Definition

An option is the right to buy (call option) or sell (put option) an asset in the future at a specific date for a particular price known as the exercise or strike price. The time period from when the option contract is written to the time when the option is exercised represents the maturity of the contract.

Key Determinants

1. The exercise price \((k)\).
2. The maturity of the contract \((h)\).
3. The discount rate \((r)\).
4. The model used to explain the evolution of stock prices over time.

The specification of the model of the underlying asset price the option contract is written on is fundamental to option pricing.

The simplest model specification (Black-Scholes) is that stock returns are normally distributed with constant volatility \(\sigma\). This is equivalent to stock prices being lognormally distributed.
For the Black-Scholes model $\sigma$ is the only unknown parameter that needs to be estimated.

1. Historical methods: use past returns of the underlying asset that the option contract is written on.
2. Implicit methods: use observed option prices.
3. Combined methods: use both returns data and option price data.

Use an iterative maximum likelihood estimator as the log-likelihood is a nonlinear function of the unknown parameter $\sigma$.

As models of price dynamics become more involved, so do the requirements placed on the estimation methods.
### Basics and Definitions

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Short-term (less than 1 year)</strong></td>
<td></td>
</tr>
<tr>
<td>European</td>
<td>Exercised on the expiration day. Payoff is $p_{t+h} - k$.</td>
</tr>
<tr>
<td>American</td>
<td>Can exercise before expiration day.</td>
</tr>
<tr>
<td>Bermudan</td>
<td>Exercise option on specific dates.</td>
</tr>
<tr>
<td>Asian</td>
<td>The payoff is based on the average of $p$ over the contract.</td>
</tr>
<tr>
<td>Barrier</td>
<td>Exercise when $p$ exceeds some “barrier”.</td>
</tr>
<tr>
<td>Binary</td>
<td>The payoff is the price of the underlying asset at maturity.</td>
</tr>
<tr>
<td><strong>Long-term (between 1 and 3 years)</strong></td>
<td></td>
</tr>
<tr>
<td>LEAPS</td>
<td>Available for equities and indexes.</td>
</tr>
<tr>
<td><strong>Options Written on More than a Single Asset</strong></td>
<td></td>
</tr>
<tr>
<td>Exchange</td>
<td>Payoff is $p_{1,t+h} - p_{2,t+h}$.</td>
</tr>
</tbody>
</table>
In determining an option contract at time \( t \), the key features are:

1. Strike or exercise price (\( k \))
   The agreed price that an investor can purchase (call) or sell (put) the underlying asset until the contract expires.

2. Maturity (\( h \))
   The time period of the contract, expressed as a proportion of a year.

3. Price of asset (\( p \))
   The price of the asset the contract is written on at the time the option is written.

The underlying asset that an option is written on can cover a broad array of asset markets including stocks, bonds, futures and foreign exchange markets.
• Option prices on March 6th, 2013, written on S&P500 index.

<table>
<thead>
<tr>
<th>Strike ($k$)</th>
<th>May Call Option Prices (c)</th>
<th>June Call Option Prices (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bid</td>
<td>Ask</td>
</tr>
<tr>
<td>1,400</td>
<td>141.40</td>
<td>144.60</td>
</tr>
<tr>
<td>1,425</td>
<td>118.80</td>
<td>120.40</td>
</tr>
<tr>
<td>1,450</td>
<td>97.10</td>
<td>100.10</td>
</tr>
<tr>
<td>1,475</td>
<td>76.20</td>
<td>79.00</td>
</tr>
<tr>
<td>1,500</td>
<td>58.00</td>
<td>59.30</td>
</tr>
<tr>
<td>1,525</td>
<td>39.90</td>
<td>41.70</td>
</tr>
<tr>
<td>1,550</td>
<td>25.20</td>
<td>26.70</td>
</tr>
<tr>
<td>1,575</td>
<td>13.90</td>
<td>15.20</td>
</tr>
<tr>
<td>1,600</td>
<td>6.70</td>
<td>7.00</td>
</tr>
<tr>
<td>1,625</td>
<td>2.90</td>
<td>3.40</td>
</tr>
<tr>
<td>1,650</td>
<td>1.15</td>
<td>1.50</td>
</tr>
<tr>
<td>1,675</td>
<td>0.35</td>
<td>0.90</td>
</tr>
<tr>
<td>1,700</td>
<td>0.10</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Source: NASDAQ Website, live quotes on March 6th, 2013.
When dealing with options data, maturity \((h)\) and volatility \((\sigma)\) are annualised. In the case of maturity

1. May contracts

\[
h(\text{May}) = \frac{25 + 30 + 18}{365} = 0.2
\]

2. For June contracts

\[
h(\text{June}) = \frac{25 + 30 + 18 + 35}{365} = 0.29589
\]

The value of the S&P500 index on March 6th in 2013 is

\[
p_t = 1,541.46
\]

“moneyness” of the contract is defined as

\[
\begin{align*}
p_t = 1,541.46 & > k : \text{in-the-money options} \\
p_t = 1,541.46 & = k : \text{at-the-money options} \\
p_t = 1,541.46 & < k : \text{out–of-the-money options}
\end{align*}
\]
Table shows that option prices increase as “moneyness” of the contract increases. This is a reflection of the payoff of the contract

\[
PAYOFF = \begin{cases} 
  p_{t+h} - k & : p_{t+h} - k \geq 0, \\
  0 & : p_{t+h} - k < 0 
\end{cases}
\]

Interpretation:

1. When \( p_{t+h} - k > 0 \), net gain from exercising the option contract in the future as the investor can purchase the S&P500 index at the price \( k \) and immediately sell the index on the spot market at the higher price of \( p_{t+h} \).
2. When \( p_{t+h} - k < 0 \), no gain to be made from exercising the contract.

But what is the future price of the asset \( p_{t+h} \)?
A comparison of the quotes for the May and June options shows that option prices are higher for the longer maturity options.

This feature of option prices is to be expected given that the S&P500 index has a positive trend suggesting that for a given strike price $k$, the longer the time period, the greater will be the expected (positive) difference between the future price $p_{t+h}$ and $k$. 
Pricing Options

- An investor at time $t$ wants to have the option to purchase an asset in the stock market in the future at $t + h$, at a price of $k$. How much should the investor be prepared to pay for the option contract now?
- To work out the price of this contract it is necessary to understand the evolution of the stock price $p_t$.
- The following Figure gives 5 potential sample paths of $p_t$ over a 6 month time horizon assuming 250 trading days in a year ($h = 0.5$), beginning at the initial spot price $p_0 = 5$. 

![Sample Paths of $p_t$](image-url)
Contracts Finishing in the Money

If $k = 4.8$, cases 1, 2 and 5 are in-the-money at $t + h$

\[
\begin{align*}
p_{1,125} & = 5.2104 > 4.8 \\
p_{2,125} & = 6.1121 > 4.8 \\
p_{5,125} & = 5.6535 > 4.8
\end{align*}
\]

Investor will exercise the option contract in either of these three cases with payoffs

\[
\begin{align*}
PAYOFF_1 & = 5.2104 - 4.8 = 0.4104 \\
PAYOFF_2 & = 6.1121 - 4.8 = 1.3121 \\
PAYOFF_5 & = 5.6535 - 4.8 = 0.8535
\end{align*}
\]
Contracts Finishing out of the Money

For cases 3 and 4 the contracts are out-of-the-money at $t + h$

\[
p_{3,125} = 4.69 < 4.8
\]
\[
p_{4,125} = 4.75 < 4.8
\]

Investor will not exercise the option contract is these two cases, so payoffs are

\[
PAYOFF_3 = 0
\]
\[
PAYOFF_4 = 0
\]
Pricing Options

- The price that the investor is prepared to pay for the option contract at the time the contract matures is the average of the payoffs

\[
\overline{PAYOFF} = \frac{1}{5} \sum_{i=1}^{5} PAYOFF_i
\]

\[
= \frac{0.4104 + 1.3121 + 0.00 + 0.00 + 0.8535}{5} = 0.5152
\]

- As the contract is written at \( t \) and not at \( t + h \), \( \overline{PAYOFF} \) at \( t + h \) is discounted back to \( t \) at the rate \( r \)

\[
c_t = \exp (-rh) \frac{1}{5} \sum_{i=1}^{5} PAYOFF_i
\]

- If \( r = 0.05 \), the price of the 6 month call option is

\[
c_t = \exp (-0.05 \times 0.5) \times 0.5152 = 0.5025
\]

so $0.5025 is the price the investor will pay for the option contract.
Allowing for $N$ time paths the option price is

$$c_t = \exp(-rh) \frac{1}{N} \sum_{i=1}^{N} PAYOFF_i$$

As the number of time paths approached infinity, $N \rightarrow \infty$, then the sample average approaches its conditional expectation in which case the option price is given by

$$c_t = \exp(-rh) E_t [PAYOFF] = \exp(-rh) E_t [p_{t+h} - k]$$

The role of Monte Carlo methods to evaluate option prices are discussed on Day 2.
Pricing Options

Specification of the Asset Price Distribution

- Important ingredients from an econometrics perspective.

1. **Forecasting**
   As options are concerned with purchasing an asset in the future at time \( t + h \), it is necessary to derive the forecast distribution of the asset price

   \[
   f(\text{p}_{t+h} | \text{p}_t)
   \]

   based on information available at time \( t \).

2. **Conditional Mean**
   The price of the asset that an investor is prepared to pay for in the future is equal to the conditional mean of \( f(\text{p}_{t+h} | \text{p}_t) \)

   \[
   \mu_{t+h|t} = \int_0^\infty \text{p}_{t+h} f(\text{p}_{t+h} | \text{p}_t) \, d\text{p}_{t+h}
   \]

3. **Truncation**
   As only value positive payoffs in the future, \( \text{p}_{t+h} - k > 0 \), the conditional mean of an option is a truncated conditional mean

   \[
   c_t = \exp(-rh) \int_k^\infty (\text{p}_{t+h} - k) f(\text{p}_{t+h} | \text{p}_t) \, d\text{p}_{t+h}
   \]
To derive the forecast distribution $f \left( p_{t+h} \mid p_t \right)$, it is necessary to specify the asset price dynamics.

The standard model is

\[
\log p_{t+h} - \log p_t = \left( r - \frac{1}{2} \sigma^2 \right) h + \sqrt{\sigma^2} h z_{t+h}
\]

\[
z_{t+h} \sim N \left( 0, 1 \right)
\]

where $h$ represents the pertinent time horizon.

The distribution of the asset price at $t + h$ conditional on information at time $t$, is a conditional lognormal distribution

\[
f \left( p_{t+h} \mid p_t \right) = \frac{1}{p_{t+h} \sqrt{2\pi\sigma^2 h}} \exp \left[ - \frac{\left( \log p_{t+h} - \mu_t \right)^2}{2\sigma^2 h} \right]
\]

where

\[
\mu_t = \log p_t + \left( r - \frac{1}{2} \sigma^2 \right) h
\]
Pricing Options

Specification of the Asset Price Distribution

- The figure gives plots of the forecast distribution for selected time horizons of 1 month \((h = 1/12)\), 3 months \((h = 3/12)\), and 6 months \((h = 6/12)\), assuming \(p_0 = 5\), \(r = 0.05\), and \(\sigma = 0.2\).
- The 1-month forecast distribution is the more compact distribution, with dispersion increasing for longer time horizons.

\[
\text{Figure:} \quad \text{Forecast distributions of the asset price for selected time horizons of 1-month (} h = 1/12 \text{), 3-months (} h = 3/12 \text{), and 6-months (} h = 6/12 \text{). Based on a log-normal distribution}\]
\[
\log \mu_t, \sigma^2 h, \text{where} \quad \mu_t = \log p_t + r \frac{1}{2} \sigma^2 h,
\]

\]
The call option price is

\[ c_t = \exp(-rh) \int_k^\infty (p_{t+h} - k) f(p_{t+h} \mid p_t) \, dp_{t+h} \]

\[ = \exp(-rh) \int_k^\infty p_{t+h} f(p_{t+h} \mid p_t) \, dp_{t+h} \]

\[ - \exp(-rh) \int_k^\infty k f(p_{t+h} \mid p_t) \, dp_{t+h} \]

\[ = \exp(-rh) l_1 - \exp(-rh) l_2 \]

For this model, analytical expressions of \( l_1 \) and \( l_2 \) are available.
Consider the first integral

\[ I_1 = \int_{k}^{\infty} p_{t+h} f\left( p_{t+h} | p_t \right) dp_{t+h} \]

\[ = \int_{k}^{\infty} p_{t+h} \times \frac{1}{p_{t+h} \sqrt{2\pi\sigma^2 h}} \exp \left[ -\frac{(\log p_{t+h} - \mu_t)^2}{2\sigma^2 h} \right] dp_{t+h} \]

\[ = \int_{k}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 h}} \exp \left[ -\frac{(\log p_{t+h} - \mu_t)^2}{2\sigma^2 h} \right] dp_{t+h} \]

Using a change of variable \( y = \log p_{t+h} \), then

\[ I_1 = \int_{\log k}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 h}} \exp \left[ -\frac{(y - \mu_t)^2}{2\sigma^2 h} \right] \exp [y] dy \]
Pricing Options

Formal Derivation

Upon simplifying (completing the square, change of variable again...)

\[ l_1 = \exp \left( \mu_t + \frac{\sigma^2 h}{2} \right) \int_{\log k}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 h}} \exp \left[ -\frac{(y - (\mu_t + \sigma^2 h))^2}{2\sigma^2 h} \right] dy \]

\[ = \exp \left( \mu_t + \frac{\sigma^2 h}{2} \right) \Phi \left( \frac{\mu_t + \sigma^2 h - \log k}{\sigma\sqrt{h}} \right) \]

where \( \Phi (a) \) is the normal cdf.
As

\[ \mu_t = \log p_t + \left( r - \frac{1}{2} \sigma^2 \right) h \]

the first integral simplifies to

\[
I_1 = \exp \left[ \log p_t + \left( r - \frac{1}{2} \sigma^2 \right) h + \frac{\sigma^2 h}{2} \right] \\
\times \Phi \left( \frac{\log (p_t / k) + \left( r + \frac{1}{2} \sigma^2 \right) h}{\sigma \sqrt{h}} \right) \\
= p_t \exp \left[ rh \right] \Phi \left( \frac{\log (p_t / k) + \left( r + \frac{1}{2} \sigma^2 \right) h}{\sigma \sqrt{h}} \right)
\]
For the second integral

\[ I_2 = \int_k^\infty f(p_{t+h} \vert p_t) \, dp_{t+h} \]

\[ = \int_k^\infty \frac{1}{p_{t+h} \sqrt{2\pi \sigma^2 h}} \exp \left[ - \frac{(\log p_{t+h} - \mu_t)^2}{2\sigma^2 h} \right] \, dp_{t+h} \]

\[ = \int_k^\infty \frac{1}{y \sqrt{2\pi \sigma^2 h}} \exp \left[ - \frac{(\log y - \mu_t)^2}{2\sigma^2 h} \right] \, dy \]

where \( y = p_{t+h} \) is used just to simplify the notation.
Pricing Options

Formal Derivation

- Using a change of variable $z = (\log y - \mu_t) / \sigma \sqrt{h}$, the integral becomes

\[
I_2 = \int_{k}^{\infty} \frac{1}{y \sqrt{2\pi \sigma^2 h}} \exp \left[ -\frac{(y - \mu_t)^2}{2\sigma^2 h} \right] dy
\]

\[
= \int_{-\infty}^{\mu_t - \log k / \sigma \sqrt{h}} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] dz
\]

\[
= \Phi \left( \frac{\mu_t - \log k}{\sigma \sqrt{h}} \right)
\]

\[
= \Phi \left( \log \left( \frac{p_t}{k} \right) + \left( r - \frac{1}{2} \sigma^2 \right) h \right)
\]

by using $\mu_t = \log p_t + \left( r - \frac{1}{2} \sigma^2 \right) h$ in the last step.
Using the expressions for $l_1$ and $l_2$ gives

$$c_t = \exp[-rh] l_1 - \exp[-rh] kl_2$$

$$= \exp[-rh] p_t \exp[rh] \Phi \left( \frac{\log \left( \frac{p_t}{k} \right) + \left( r + \frac{1}{2} \sigma^2 \right) h}{\sigma \sqrt{h}} \right)$$

$$- \exp[-rh] k \Phi \left( \frac{\log \left( \frac{p_t}{k} \right) + \left( r - \frac{1}{2} \sigma^2 \right) h}{\sigma \sqrt{h}} \right)$$

$$= p_t \Phi \left( \frac{\log \left( \frac{p_t}{k} \right) + \left( r + \frac{1}{2} \sigma^2 \right) h}{\sigma \sqrt{h}} \right)$$

$$- \exp[-rh] k \Phi \left( \frac{\log \left( \frac{p_t}{k} \right) + \left( r - \frac{1}{2} \sigma^2 \right) h}{\sigma \sqrt{h}} \right)$$
A more compact expression for the call option is

\[
c_t = p_t \Phi (d) - ke^{-rh} \Phi (d - \sigma \sqrt{h})
\]

where

\[
d = \frac{\log (p_t / k) + (r + \frac{1}{2} \sigma^2) h}{\sigma \sqrt{h}}
\]

This is the Black-Scholes price of a European call option contract.
The Black-Scholes Model

- Black and Scholes (1973) derived a model of financial markets allowing for European derivative securities.
- Various methods exist to solve for the Black-Scholes price, including methods based on stochastic calculus with the price given as the solution of a partial differential equation.
- The solution of the price given above is based on solving for the mean of a truncated distribution where truncation is based on discontinuous payoff function at $k$.
- Advantages
  1. Highlights that the price is a function of a (truncated) conditional mean of the asset price distribution at the time of maturity.
  2. Suggests how the model can be expanded.
The Black-Scholes Model
Call Options

- An option price is equal to the expected value of its discounted payoff.
- For a European call option expiring at \( t + h \), the price is

\[
c_t = e^{-rh} E_t \left[ \max (p_{t+h} - k, 0) \right]
\]

- Assuming that \( p_t \) is lognormally distributed gives

\[
c_t = p_t \Phi(d) - ke^{-r_th} \Phi(d - \sigma \sqrt{h})
\]

where

\[
d = \frac{\log(p_t/k) + (r + \frac{1}{2} \sigma^2)h}{\sigma \sqrt{h}}
\]

and \( \Phi(z) \) is the standard normal cumulative distribution.
The Black-Scholes Model

Call Options

Interpretation

1. The term $\Phi(d - \sigma \sqrt{h})$ represents the probability the option is exercised (in a risk neutral world) so $k \Phi(d - \sigma \sqrt{h})$ is the strike price times the probability that the strike price is paid.

2. The term $p_t \Phi(d)$ is the discounted expected value that the price at maturity is $p_{t+h}$, provided that $p_{t+h} > k$.

3. The plus sign in front of $p_t \Phi(d)$, and a negative sign in front of, suggests the investor takes a long position in holding the underlying asset and a short position in holding the option.
Special Cases

1. If the distribution is not truncated so \( k = 0 \), then \( d \to \infty \), whereby

\[
\lim_{k \to 0} \Phi(d) \to 1
\]

resulting in

\[
c_t = p_t
\]

2. Alternatively, if \( k = 0 \), the conditional mean of the price at \( t + h \) for a lognormal random variable is simply \( p_t \exp(rh) \), which reduces to \( p_t \) when discounted back to \( t \), when the contract is written using \( \exp(-rh) \) as the discount factor.
The Black-Scholes Model

Call Options

Example (Pricing a Call Option)

Consider an option contract that expires in 3 months, with $k = \$6.00$. If $r = 0.059$ and $\sigma = 0.15$, the price of the equity at $t$ is $p_t = \$6.20$, then

$$d = \frac{\log(p_t / k) + (r + \frac{1}{2} \sigma^2) h}{\sigma \sqrt{h}} = 0.67136$$

Now

$$\Phi(d) = \Phi(0.67136) = 0.7490, \quad \Phi(d - \sigma \sqrt{h}) = \Phi(0.5964) = 0.7245$$

so the price of a call option is

$$c_t = p_t \Phi(d) - ke^{-rh} \Phi(d - \sigma \sqrt{h})$$

$$= 6.20 \times 0.7490 - 6.00 \times \exp(-0.0509 \times 0.25) \times 0.7245$$

$$= 0.3518$$
For a European put option expiring at $t + h$, the price is

$$put_t = e^{-rh} E_t \left[ \max (k - p_{t+h}, 0) \right]$$

Assuming that $p_t$ is lognormally distributed gives

$$put_t = ke^{-r_t h} \Phi(-d + \sigma \sqrt{h}) - p_t \Phi(-d)$$

where

$$d = \frac{\log(p_t/k) + (r + \frac{1}{2} \sigma^2) h}{\sigma \sqrt{h}}$$

and $\Phi(z)$ is the standard normal cumulative distribution.
The Black-Scholes Model

Stock Options

- To allow for options written on stocks earning dividends (equal to a continuous dividend stream at the rate \( q \) per annum), the asset price equation becomes

\[
\log p_{t+h} - \log p_t = (r - q + \frac{1}{2} \sigma^2) h + \sqrt{\sigma^2 h} z_{t+h}
\]

\[z_{t+h} \sim N(0, 1)\]

- The Black-Scholes price for a European call option paying a continuous dividend yield, is

\[
c_t = p_t e^{-qh} \Phi(d) - ke^{-rh} \Phi(d - \sigma \sqrt{h})
\]

where now

\[
d = \frac{\log(p_t / k) + (r - q + \frac{1}{2} \sigma^2) h}{\sigma \sqrt{h}}
\]
The Black-Scholes Model

Currency Options

- Let $p_t$ represent the exchange rate between two currencies.
- The asset exchange rate equation becomes

\[
\log p_{t+h} - \log p_t = (r - i - \frac{1}{2} \sigma^2) h + \sqrt{\sigma^2 h} z_{t+h}
\]

\[
z_{t+h} \sim N(0, 1)
\]

where $r$ is the domestic interest rate, and $i_t$ is the foreign interest rate.

- Equation represents uncovered interest rate parity as $E_t [\log p_{t+h} - \log p_t]$ measures the expected depreciation of the exchange rate, which equals $r - i$, the interest rate differential between two countries.

- The Black-Scholes price for a European currency call option is

\[
c_t = p_t e^{-ih} \Phi(d) - ke^{-rh} \Phi(d - \sigma \sqrt{h}),
\]

where now

\[
d = \frac{\log(p_t/k) + (r - i + \frac{1}{2} \sigma^2) h}{\sigma \sqrt{h}}
\]
An exchange option provides the right to exchange one asset for another asset - taken as exchange asset 2 for asset 1.

The holder of the exchange option is betting that the price of asset 1 will rise relative to asset 2.

For a European exchange option which gives the holder the right to exchange asset 2 for asset 1 when the contract expires at time $t + h$, the price of an exchange option ($c_t$) is given by the expected value of its discounted payoff

$$c_t = e^{-rh} E_t \left[ \max (p_{1,t+h} - p_{2,t+h}, 0) \right]$$

where $p_{i,t}$ is the price of the $i^{th}$ asset.

As both prices are assumed to be stochastic, the call option is effectively a model with a stochastic strike price.
The asset price dynamics are

\[
\begin{align*}
\log p_{1,t+h} - \log p_{1,t} &= (r - q_1 - \frac{1}{2}\sigma_1^2) h + \sqrt{\sigma_1^2} h z_{1,t+h} \\
\log p_{2,t+h} - \log p_{2,t} &= (r - q_2 - \frac{1}{2}\sigma_2^2) h + \sqrt{\sigma_2^2} h z_{2,t+h}
\end{align*}
\]

where

\[
\begin{bmatrix}
z_{1,t} \\
z_{2,t}
\end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix}
0 \\
0
\end{bmatrix}, \begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix} \right)
\]

The Black-Scholes price is

\[
c_t = p_{1,t} e^{-q_1 h} N(d) - p_{2,t} e^{-q_2 h} N\left(d - \sigma \sqrt{h}\right),
\]

where

\[
d = \frac{\ln \left(p_{1,t} / p_{2,t}\right) + (q_2 - q_1 + \sigma^2 / 2) (T - t)}{\sigma \sqrt{T - t}}
\]

and

\[
\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2}.
\]
A First Look at the Data

The data consist of \( N = 27,695 \) contracts on European call options written on the S&P500 stock index on the 4th of April, 1995.

In selecting the options data, three filters are applied.

1. Option prices satisfy the no arbitrage lower bound condition

\[
LB = \max \left( p_t e^{-q_h} - k e^{-r_h}, 0 \right)
\]

2. Include only contracts conducted between 9am and 3pm to minimize nonsynchronicity problems.

3. Duplicate observations are removed to ensure that observations are unique.
A First Look at the Data

The Table summarizes the key characteristics of European call option data.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Option Price ((c_t))</th>
<th>Strike Price ((k))</th>
<th>Maturity ((h))</th>
<th>Stock Price ((p_t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>27,695</td>
<td>30</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Minimum</td>
<td>$1.0350</td>
<td>$350.0000</td>
<td>0.123288</td>
<td>$496.4390</td>
</tr>
<tr>
<td>Maximum</td>
<td>$156.2500</td>
<td>$550.0000</td>
<td>0.457534</td>
<td>$502.9450</td>
</tr>
<tr>
<td>Mean</td>
<td>$60.36025</td>
<td>$449.2538</td>
<td>0.270635</td>
<td>$499.8471</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>$30.97777</td>
<td>$33.8460</td>
<td>0.141486</td>
<td>$1.9290</td>
</tr>
</tbody>
</table>
The call option prices \( (c_t) \) are computed as the average of the bid–ask prices. The average price is $60.3602 and a standard deviation of $30.9778.

There are 30 different strike prices \( (k) \) with 3 different maturities resulting in a total of \( 30 \times 3 = 90 \) unique contracts for the day. Strike price ranges from $350 to $550.

The 3 different maturities are May, June and September.

The stock price \( (p_t) \) is the synchronously recorded price of the index at the time the quote is recorded.

The stock price is adjusted for dividends by multiplying the price of the S&P500 stock index by \( e^{-q_h} \), where \( q \) is computed as the average annual rate of dividends paid on the S&P500 index over 1995. The dividend adjusted stock price varies over the day with an average price of $499.8471 ranging from $496.4390 to $502.9450.

The interest rate \( (r) \) is the annualized 3-month Treasury bill rate on April 4, which is set at 5.61% over the trading day.
A First Look at the Data

The following table gives the first 13 contracts in the data set.

<table>
<thead>
<tr>
<th>Contract Number</th>
<th>Option Price ($c_t$)</th>
<th>Strike Price ($k$)</th>
<th>Maturity ($h$)</th>
<th>Stock Price ($p_t$)</th>
<th>Interest Rate ($r$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7550</td>
<td>550</td>
<td>September</td>
<td>496.8740</td>
<td>0.0591</td>
</tr>
<tr>
<td>2</td>
<td>4.9400</td>
<td>535</td>
<td>September</td>
<td>496.7260</td>
<td>0.0591</td>
</tr>
<tr>
<td>3</td>
<td>4.9400</td>
<td>535</td>
<td>September</td>
<td>497.0910</td>
<td>0.0591</td>
</tr>
<tr>
<td>4</td>
<td>4.8800</td>
<td>535</td>
<td>September</td>
<td>497.1010</td>
<td>0.0591</td>
</tr>
<tr>
<td>5</td>
<td>5.0650</td>
<td>535</td>
<td>September</td>
<td>498.2770</td>
<td>0.0591</td>
</tr>
<tr>
<td>6</td>
<td>6.3800</td>
<td>530</td>
<td>September</td>
<td>496.9830</td>
<td>0.0591</td>
</tr>
<tr>
<td>7</td>
<td>6.3150</td>
<td>530</td>
<td>September</td>
<td>497.0220</td>
<td>0.0591</td>
</tr>
<tr>
<td>8</td>
<td>6.3150</td>
<td>530</td>
<td>September</td>
<td>497.0520</td>
<td>0.0591</td>
</tr>
<tr>
<td>9</td>
<td>6.8750</td>
<td>530</td>
<td>September</td>
<td>498.5040</td>
<td>0.0591</td>
</tr>
<tr>
<td>10</td>
<td>6.8150</td>
<td>530</td>
<td>September</td>
<td>498.5040</td>
<td>0.0591</td>
</tr>
<tr>
<td>11</td>
<td>1.0950</td>
<td>525</td>
<td>May</td>
<td>500.9720</td>
<td>0.0591</td>
</tr>
<tr>
<td>12</td>
<td>2.5650</td>
<td>525</td>
<td>June</td>
<td>499.9930</td>
<td>0.0591</td>
</tr>
<tr>
<td>13</td>
<td>1.0650</td>
<td>525</td>
<td>May</td>
<td>501.0410</td>
<td>0.0591</td>
</tr>
</tbody>
</table>
For the Black-Scholes option price model the only unknown parameter is the volatility parameter \((\sigma)\), corresponding to the volatility of the returns on the stock price that the option contract is written on.

All remaining components used to price an option are specified by the contract.

There are three broad methods to estimate \(\sigma\).

1. Historical methods: use past returns of the underlying asset that the option contract is written on.
2. Implicit methods: use observed option prices.
3. Combined methods: use both returns data and option data.
As $\sigma$ in the Black-Scholes European call options model corresponds to the standard deviation of stock returns, a natural estimator of $\sigma$ is then obtained by computing the standard deviation of stock returns. As $\sigma$ needs to be expressed in annual terms, this estimate needs to be scaled. This approach is commonly referred to as the historical method. Suppose that stock prices ($p_t$) are measured daily.
The steps to compute $\sigma$, are as follows.

1. First, compute daily returns from the stock prices as

$$y_t = \log(p_t) - \log(p_{t-1})$$

For a sample of $T$ daily observations on returns, now compute an estimate of the variance of the daily stock returns as

$$\hat{\sigma}_y^2 = \frac{1}{T} \sum_{t=1}^{T} (y_t - \bar{y})^2$$

where $\bar{y}$ is the sample mean of $y_t$.

2. An estimate of the annualized variance is then

$$\hat{\sigma}^2 = 250\hat{\sigma}_y^2,$$

where 250 is chosen to represent the number of trading days that exist within a calendar year.

3. An estimate of the annualized volatility parameter is

$$\hat{\sigma} = \sqrt{250\hat{\sigma}_y}$$
Example (Estimating Volatility by Stock Prices)

Daily data on the S&P500 stock index for 1995, from January 2 to December 31, are given in the Figure.

Daily log-returns computed as $y_t = \log p_t - \log p_{t-1}$, which begin January 3, resulting in a sample size of $T = 259$. 
Example (Estimating Volatility by Stock Prices continued)

The mean of daily log-returns is \( \bar{r} = 0.001133 \) while the estimate of the variance is

\[
\hat{\sigma}^2_y = \frac{1}{T-1} \sum_{t=1}^{T} (y_t - \bar{y})^2 = \frac{0.006067}{259 - 1} = 2.3516 \times 10^{-5}
\]

so

\[
\hat{\sigma}_y = \sqrt{2.3516 \times 10^{-5}} = 4.8493 \times 10^{-3}
\]

The annualized estimate of volatility in stock returns is

\[
\hat{\sigma} = \sqrt{250 \times \hat{\sigma}^2_y} = \sqrt{250 \times 2.3516 \times 10^{-5}} = 7.6675 \times 10^{-2}
\]

or 7.6675% per annum.
Example (Estimating Volatility by Stock Prices continued)

Alternatively, expressing the annualized volatility in terms of calendar days the estimate becomes

\[
\sqrt{365 \times 2.3516 \times 10^{-5}} = 9.2646 \times 10^{-2}
\]

Or, using just returns in May

\[
\hat{\sigma}_y = 0.006572
\]

yielding an annualized estimate of volatility of based on calendar days gives

\[
\hat{\sigma} = \sqrt{365} \times \hat{\sigma}_y = \sqrt{365} \times 0.006572 = 0.12556
\]

or 12.556% per annum.
Constant Mean Model with Constant Volatility

- An alternative approach is to estimate the “constant mean” model of returns given by

\[ \log p_t - \log p_{t-1} = \gamma + u_t \]

\[ u_t \sim iid \ N(0, \sigma_y^2) \]

where \( \gamma \) is mean return and \( \sigma_y \) is the daily “historical” volatility.

- The log-likelihood function is

\[
\log L = -\frac{T}{2} \log (2\pi \sigma_y^2) - \frac{1}{2} \sum_{t=1}^{T} \left( \frac{\log p_t - \log p_{t-1} - \gamma}{\sigma_y} \right)^2
\]

where the unknown parameters are

\[ \theta = \{ \gamma, \sigma_y^2 \} \]
Constant Mean Model with Constant Volatility

- Maximising the log-likelihood function is achieved in one step with the maximum likelihood estimators given by

\[
\hat{\gamma} = \frac{1}{T} \sum_{t=1}^{T} (\log p_t - \log p_{t-1})
\]

\[
\hat{\sigma}^2_y = \frac{1}{T} \sum_{t=1}^{T} (\log p_t - \log p_{t-1} - \hat{\gamma})^2
\]

- The MLEs of \( \gamma \) and \( \sigma^2_y \) are also the OLS estimators as MLE and OLS are equivalent in this case.
Example (Estimating Volatility by Stock Prices)

Using the $T = 259$ observations on the S&P500 index, the sample becomes 258 from computing returns. The log-likelihood is

$$\log L(\theta) = -\frac{T}{2} \log (2\pi \sigma_y^2) - \frac{1}{2} \sum_{t=1}^{T} \left( \frac{\log p_t - \log p_{t-1} - \gamma}{\sigma_y} \right)^2$$

where the unknown parameters are $\theta = \{\gamma, \sigma_y^2\}$. The maximum likelihood estimates are

$$\hat{\gamma} = 0.001133, \quad \hat{\sigma}_y^2 = 0.004849^2$$

In which case the estimate of volatility is

$$\hat{\sigma} = \sqrt{250\hat{\sigma}_y^2} = \sqrt{250 \times 0.004849^2} = 7.6669 \times 10^{-2}$$

which is the same value as based on descriptive statistics.
An alternative approach is to estimate the “constant mean” model with a GARCH conditional variance

\[
\log p_t - \log p_{t-1} = \gamma + u_t \\
\sigma_{y,t}^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{y,t-1}^2 \\
u_t \sim iid N(0, \sigma_{y,t}^2)
\]

where \(\gamma\) is mean return and \(\sigma_{y,t}^2\) is the GARCH conditional variance.

The variance is now time-varying with the “memory” controlled by the parameters \(\alpha_1\) and \(\beta_1\).
The log-likelihood function is

\[
\log L = -\frac{T}{2} \log (2\pi) - \frac{1}{2} \sum_{t=1}^{T} \sigma_{y,t}^2 - \frac{1}{2} \sum_{t=1}^{T} \left( \log p_t - \log p_{t-1} - \gamma \right)^2 \sigma_{y,t}^2
\]

\[
= -\frac{T}{2} \log (2\pi) - \frac{1}{2} \sum_{t=1}^{T} \left( \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{y,t-1}^2 \right)
\]

\[
- \frac{1}{2} \sum_{t=1}^{T} \left( \log p_t - \log p_{t-1} - \gamma \right)^2 \sigma_{y,t-1}^2
\]

where the unknown parameters are

\[
\theta = \{ \gamma, \alpha_0, \alpha_1, \beta_1 \}
\]

Maximising the log-likelihood function requires an iterative solution as the log-likelihood is a nonlinear function of the parameters.
Example (Estimating Volatility by Stock Prices)

Using the $T = 259$ observations on the S&P500 index, the sample becomes 258 from computing returns. The log-likelihood is

$$\log L = -\frac{T}{2} \log (2\pi) - \frac{1}{2} \sum_{t=1}^{T} \left( \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{y,t-1}^2 \right)$$

$$- \frac{1}{2} \sum_{t=1}^{T} \left( \log p_t - \log p_{t-1} - \gamma \right)^2$$

$$- \frac{1}{2} \sum_{t=1}^{T} \frac{\alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{y,t-1}^2}{\alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{y,t-1}^2}$$

where the unknown parameters are $\theta = \{ \gamma, \alpha_0, \alpha_1, \beta_1 \}$.
Example (Estimating Volatility by Stock Prices continued)

The maximum likelihood estimates are

\[ \hat{\gamma} = 0.001144, \quad \hat{\alpha}_0 = 1.29 \times 10^{-6}, \quad \hat{\alpha}_1 = 0.0128, \quad \hat{\beta}_1 = 0.9357 \]

An estimate of the long-run variance from the GARCH estimates is

\[ \hat{\sigma}_y^2 = \frac{\hat{\alpha}_0}{1 - \hat{\alpha}_1 - \hat{\beta}_1} = \frac{1.29 \times 10^{-6}}{1 - 0.0128 - 0.9357} = 2.5049 \times 10^{-5} \]

The annualised volatility based on the long-run GARCH estimate is

\[ \hat{\sigma} = \sqrt{250} \times 2.5049 \times 10^{-5} = 3.9606 \times 10^{-4} \]

This estimate is smaller than the estimate based on the descriptive statistics.
Example (Estimating Volatility by Stock Prices continued)

The following figure highlights the reason why the volatility estimates based on GARCH are smaller as the estimates of the volatility are relatively low in the first part of the period, which eventually climb to about 0.005, an estimate consistent with the constant variance models.
Given \( i = 1, 2, \cdots, N \) observations on the observed market prices of option contracts
\[
\{ c_1, c_2, \cdots, c_N \}
\]
and the corresponding time-stamped price
\[
\{ p_1, p_2, \cdots, p_N \}
\]
of the underlying asset the option is written on, it is possible to back-out an estimate of the volatility \( \sigma \).

This estimate represents an implied estimate of \( \sigma \) as it is the implied value of volatility that must have been used to price the option in the first place.
The empirical model is

\[ c_i = c_i^{BS}(\sigma) + u_i \]

where \( c_i \) represents the observed price of the \( i^{th} \) option and \( c_i^{BS}(\sigma) \) is the Black-Scholes price of the \( i^{th} \) contract

\[ c_i^{BS} = p_i \Phi(d_i) - k_i e^{-r_i h_i} \Phi(d_i - \sigma \sqrt{h_i}) \]

with \( d_i \) defined as

\[ d_i = \frac{\log(p_i / k_i) + (r_i + \frac{1}{2} \sigma^2) h_i}{\sigma \sqrt{h_i}} \]

The subscript \( i \) on \( \{p_i, k_i, h_i, r_i\} \) allows for different option contracts.
The disturbance term

\[ u_i = c_i - c_i^{BS}(\sigma) \]

represents the pricing error which is assumed to satisfy

\[ u_i \sim iid \ N \left(0, \omega^2 \right) \]
To estimate the parameter

\[ \theta = \{ \sigma^2, \omega^2 \} \]

of the empirical model by maximum likelihood, the logarithm of the likelihood is given by

\[
\log L = - \frac{N}{2} \log (2\pi \omega^2) - \frac{1}{2} \sum_{i=1}^{N} \left( \frac{c_i - c_i^{BS}(\sigma)}{\omega} \right)^2
\]

where \( N \) is the number of option contracts in the sample.

As \( c_i^{BS}(\sigma) \) is a nonlinear function of \( \sigma \) an iterative algorithm is needed to estimate the parameters \( \theta \).
Example (Estimating Volatility Using Option Prices)

Using the $N = 27,695$ option contracts on April 4, 1995, the log-likelihood is

$$\log L = -\frac{N}{2} \log (2\pi \omega^2) - \frac{1}{2} \sum_{i=1}^{N} \left( \frac{c_i - c_i^{BS} (\sigma)}{\omega} \right)^2$$

where the unknown parameters are $\theta = \{ \sigma^2, \omega^2 \}$

$$c_i^{BS} = p_i \Phi (d_i) - k_i e^{-r h_i} \Phi (d_i - \sigma \sqrt{h_i})$$

$$d_i = \frac{\log (p_i / k_i) + (r + \frac{1}{2} \sigma^2) h_i}{\sigma \sqrt{h_i}}$$

The starting estimates are $\theta (0) = \left\{ \sigma^2 (0) = 0.1, \omega^2 (0) = 0.1 \right\}$.
Example (Estimating Volatility Using Option Prices continued)

The number of iterations for the algorithm to converge is 59 yielding a log-likelihood value of

$$\log L (\hat{\theta}) = 38217.69$$

The parameter estimates are

$$\hat{\theta} = \left\{ \hat{\sigma}^2 = 0.017024, \hat{\omega}^2 = 0.924947 \right\}$$

with standard errors of $$se(\hat{\sigma}^2) = 2.98 \times 10^{-5}$$ and $$se(\hat{\omega}^2) = 0.7975 \times 10^{-2}$$ respectively. The (implied) volatility estimate is then

$$\hat{\sigma} = \sqrt{0.017024} = 0.1305$$

or 13.05%. This estimate is roughly twice the size of the volatility estimate based on the stock price data given in previous example.
An alternative approach is to use information on both the returns of the underlying asset that the option is written on as well as the prices of the option.

There are two ways to include option prices.

1. Direct Approach: use data on observed option prices, \( c_t \), in the case of call options.
2. Indirect Approach: use data on the VIX.

The use of the VIX is motivated by the property that it approximates the 30-day variance swap rate on the S&P500 index, thereby providing a measure of the risk-neutral expectation of integrated variance over a month (Bollerslev, Gibson and Zhou (2011))

\[
\left( \frac{VIX_t}{100} \right)^2 = \frac{365}{30} E_t^* \sum_{j=1}^{30} h_{t+j}
\]

where \( E_t^* \) is the expectation under the risk neutral measure.
Consider the following variation of the constant mean model with GARCH conditional variance

\[
\log p_t - \log p_{t-1} = r + \lambda \sigma_{y,t} - \frac{1}{2} \sigma_{y,t}^2 + \omega_t z_t
\]

\[
\sigma_{y,t}^2 = \alpha_0 + (\alpha_1 z_{t-1}^2 + \beta_1) \sigma_{y,t-1}^2
\]

\[z_t \sim iid \ N(0, 1)\]

where \( r \) is the risk-free rate of interest.

The return dynamics expressed under the risk-neutral measure is

\[
\log p_t - \log p_{t-1} = r - \frac{1}{2} \sigma_{y,t}^2 + \omega_t z_t^*
\]

\[
\sigma_{y,t}^2 = \alpha_0 + (\alpha_1 (z_{t-1}^* - \lambda)^2 + \beta_1) \sigma_{y,t-1}^2
\]

\[z_t^* \sim iid \ N(0, 1)\]

where \( z_t^* = z_t + \lambda \) is the risk-neutral probability measure.
For the variance to be weakly stationary

$$\psi = \alpha_1 (1 + \lambda^2) + \beta_1 < 1$$

which represents the persistence under the risk-neutral probability measure.

Given initial values for the parameters and an initial value $\sigma_{y,0}^2$, from the expression of the GARCH variance it is possible to derive estimates of the conditional variance from the returns data as

$$\sigma_{y,t+1}^2 = \alpha_0 + (\alpha_1 z_t^2 + \beta_1) \sigma_{y,t}^2$$

where $z_t$ are the standardised returns given by

$$z_t = \frac{\log p_t - \log p_{t-1} - (r + \lambda \sigma_{y,t} - \frac{1}{2} \sigma_{y,t}^2)}{\sigma_{y,t}}$$
Estimation
Combining Returns and Options Price Data

Now

$$E_t^* \sigma_{y,t+n}^2 = \sigma_{y,t+1}^2 \psi^{n-1} + \alpha_0 \sum_{i=1}^{n-1} \psi^{i-1}$$

where

$$\psi = \alpha_1 (1 + \lambda^2) + \beta_1$$

In which case

$$VIX_t = 100 \times \sqrt{\frac{365}{30} \sum_{j=1}^{30} \left( \sigma_{y,t+1}^2 \psi^{n-1} + \alpha_0 \frac{1 - \psi^{j-1}}{1 - \psi} \right)}$$

This expression provides an alternative estimate of the variance based on $VIX_t$ given by

$$\sigma_{y,t}^* = \frac{1}{\sum_{j=1}^{30} \psi^{j-1}} \left( \frac{30}{365} \left( \frac{VIX_t}{100} \right)^2 - \alpha_0 \sum_{j=1}^{30} \frac{1 - \psi^{j-1}}{1 - \psi} \right)$$
The two alternative expressions for the conditional variances suggests that for \( T \) observations on returns and the VIX, the parameters of the model can be estimated by solving

\[
\hat{\theta} = \arg \min_{\theta} \frac{1}{T} \sum_{t=1}^{T} \left( \sigma_{y,t}^* - \sigma_{y,t}^2 \right)^2
\]

where the unknown parameters are

\[
\theta = \{ \alpha_0, \alpha_1, \beta_1, \lambda \} \]