Optimal Self-Enforcement and Termination

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Abstract

We study a dynamic principal-agent problem where the agent receives a stochastic outside opportunity/offer each period and he cannot commit to not leaving the ongoing relationship. Termination, while costly, allows the principal to go back to an external market to hire a new agent. We treat self-enforcement and termination both as endogenous variables and let the principal respond strategically to the agent’s outside offers. The optimal contract is either dynamic or stationary. If it starts with a sufficiently low expected utility for the agent, then the principal matches the agent’s outside offer to retain him whenever the value of the outside offer is above his current expected utility but below a constant cutoff for termination, and the agent is terminated whenever the value of the outside offer exceeds that constant cutoff. If the optimal contract starts the agent with a sufficiently high expected utility, then it is stationary: the agent’s expected utility is constant conditional on retention, and he is terminated once he receives an outside offer higher than a constant cutoff for termination. The stationary optimal contract generates both voluntary and involuntary terminations. All dynamic optimal contracts converge stochastically but monotonically to a single stationary optimal contract.
1 Introduction

Models of dynamic contracts with limited commitment (e.g., Thomas and Worrall, 1988; Phelan, 1995; Kocherlakota, 1996) study problems of optimal contracting in environments where one or more contracting parties of the relationship are exposed to outside opportunities that generate temptations for them to leave the ongoing relationship. The solution concept the existing models all use is to impose universal self-enforcement: the contract must be designed to be such that in all ex post states of the world the agents never have incentives to leave the ongoing relationship which, therefore, lasts forever.

In this paper, we study a model that has in essence the same economic tension that the models in the above literature possess, but we approach the problem differently. Instead of imposing universally a self-enforcing constraint in each and every state of the world that the contract may dictate, we let the contract choose, optimally, whether or not to impose such a constraint in each individual state of the world. In other words, we make self-enforcement endogenous, by choosing optimally a subset of the states of the world to impose the self-enforcing constraint, while letting the contract terminate (letting the agent pursue the outside opportunity he receives) in the states outside of this subset.

Specifically, we study a problem of dynamic contracting between a risk neutral principal and a risk averse agent where each period a stochastic outside opportunity (offer), drawn independently from a given distribution, arises for the agent. There is less than full commitment: the agent in each period is free to walk away from the ongoing contract to pursue the outside offer he receives. Thus, upon the realization of any specific outside opportunity, the contract must decide whether the agent should take it, in which case the current relationship must be terminated; or not to take it, in which case the current relationship continues, but the continuation of the contract must give the agent a value higher than his current outside offer – the contract must be self-enforcing. Termination is costly: there is a fixed cost that the principal must incur each time termination occurs. Upon any termination, the agent leaves to pursue the outside opportunity he receives, and the principal then goes back to an external market to find and start a new relationship with a new agent.

Suppose the cost of termination is infinitely high and so termination is never optimal. Then in order to enforce continuation - to cope with the problem of the lack of commitment - the optimal contract must match the agent’s outside offer whenever it exceeds his current promised utility, pushing up monotonically the agent’s expected utility over time. This results in an efficiency loss, as optimality under full commitment would dictate perfect intertemporal consumption smoothing and a time-invariant expected utility for the agent.
When the cost of termination is finite, termination not only allows the principal to utilize the outside offers as an external means for compensating the agent, but can also be used as a device for mitigating the effects of the lack of commitment. Specifically, in any state where the lack of commitment may be a binding constraint, instead of imposing a self-enforcing constraint in that state (matching the agent’s outside offer to induce him to stay), the contract could choose to terminate the agent. Termination undoes the effects of the lack of commitment, allowing the principal to avoid the cost of self-enforcement in that state. It then follows that higher outside offers generate stronger incentives for termination, since the agent receiving a higher outside offer must be promised a higher utility in order to induce him to stay. Higher outside offers also make termination more worthwhile because of the higher values the agent receives upon termination. With the optimal contract then, sufficiently high outside offers always induce termination. It also follows that a higher expected utility for the agent dictates less termination. This is because the problem of the lack of commitment diminishes as the agent’s expected utility increases, generating weaker demand for using termination as a device for alleviating the effects of the lack of commitment. Higher expected utilities for the agent require higher utilities be promised to the agent in all states of his outside offers, relaxing the self-enforcing constraint.

With the optimal contract, a binding self-enforcing constraint is imposed in a state of the agent’s outside offer (the principal matches the agent’s outside offer) if and only if the offer is neither sufficiently high to warrant termination, nor sufficiently low (below the agent’s current expected utility) to make retention voluntary under full commitment. Since the problem of the lack of commitment diminishes as the agent’s expected utility increases, the size of the set of the agent’s outside offers where self-enforcement is imposed diminishes as the agent’s expected utility increases. Moreover, for sufficiently high levels of the agent’s expected utility, the self-enforcing constraint is not imposed in any of the states of the agent’s outside offer – the principal never matches the outside offers of an agent with a sufficiently high expected utility. For the agent with a sufficiently high expected utility, termination alone is sufficient for tackling the problem of the lack of commitment.

The model thus produces interesting dynamics and stationarity properties for the optimal contract, depending on the agent’s starting expected utility. Suppose the contract starts the agent with a sufficiently low expected utility. Then each time he receives an outside offer above his current promised utility but below a constant upper bound, the principal matches his outside offer to retain him; otherwise he is either let go to pursue an outside offer that is strictly better than his current contract, or he and the principal would simply disregard the outside offer. Thus, conditional on that the agent stays with the principal, over time his expected utility increases and converges stochastically to the constant upper bound for retention. On the path of convergence,
higher and higher outside offers are matched, with lower and lower probabilities. In the limit, that is, at the upper bound for retention, the optimal contract is such that the agent stays whenever his outside offer is below his constant value of continuation, and quits otherwise; and the principal never reacts to the agent’s outside offer. Suppose the contract starts the agent with a sufficiently high expected utility. Then the optimal contract is stationary: the agent’s expected utility is constant conditional on retention, and he is terminated whenever he receives an outside offer higher than a constant cutoff for termination.

When the optimal contract is stationary, it generates both voluntary and involuntary terminations, as the cutoff for termination is strictly below the agent’s value of continuation with the current contract. Specifically, the agent is terminated involuntarily after receiving an outside offer above the cutoff for termination but below the value of continuation with the current contract. The agent is terminated voluntarily if he receives an outside offer higher than the value of his current contract.

Because termination may help mitigate the problem of the lack of commitment, a lower termination cost, which facilitates more efficient use of termination as a device for mitigating limited commitment, should give rise to less efficiency losses from the lack of commitment. Specifically, compared to the case where termination is infinitely costly, the optimal contract in the case where termination is feasible at a finite cost dictates a smaller set of the states in which a binding self-enforcing constraint must be imposed for optimality. Moreover, the size of this set shrinks monotonically and converges to zero as the cost of termination decreases and converges to zero.

One way to interpret our model is to view it as a model of on-the-job search. The job is a dynamic contract between a risk neutral firm and a risk averse worker. The search is a sequence of stochastic draws of outside opportunities that the employed worker receives. The worker is free to walk away from the firm to pursue any outside offer he receives - limited commitment. Termination is feasible but costly to the firm. After terminating an incumbent worker, the firm goes back to an external labor market to hire a new worker. In such a setting of on-the-job search we fully characterize the dynamics of the employment relationship. In particular, we show that the optimal contract permits three and only three ways in which the firm responds to the worker’s outside offer. One, the firm does not respond to the worker’s outside offer (offering him the same continuation contract) but the worker chooses to stay, voluntarily, with the firm. Two, retain the worker by matching his outside offers. Three, terminate the worker by letting him quit voluntarily, or involuntarily. In an involuntary termination, the worker must accept an outside offer that is inferior to his current contract.
Most existing models of on-the-job search do not model how the firm responds optimally to the worker’s current outside offers. In Burdett (1978), for example, the employment contract is governed by a fixed wage while on-the-job search is a sequence of costly random draws from a given distribution of outside wage offers. Separation (voluntary quit) occurs whenever the worker receives an outside offer that is better than the fixed wage. The recent important work of Burdett and Coles (2003) studies an equilibrium labor market where labor contracts are designed efficiently with respect to the worker’s tenure with the firm, but not to his outside opportunities. On-the-job search then implies it is optimal for the principal to backload wages over the agent’s tenure, and this generates dynamics within the firm.

The few existing papers that model more explicitly the interaction between the firm that currently employs the worker and the firm that’s making the external offer take a more game theoretic approach. Moscarini (2005) analyzes the role of on-the-job search for job separation and equilibrium wage distribution in a Jovanovic (1984) type learning model that is nested in the Mortensen and Pissarides (1994) framework of search and matching. There, when an employed worker has a successful contact with an outside vacancy, the incumbent firm and the poaching firm play a sequential auction game to determine a unique outcome for the worker: either the incumbent firm does not respond to the outside offer and the worker quits, or the incumbent firm and the worker disregard the outside offer. In Moscarini (2005), unlike in our model, the firm never matches the worker’s outside offer.

Postel-Vinay and Robin (2002) constructs and estimates an equilibrium model of search and on-the-job search to generate equilibrium wage dispersion with heterogeneous firms and workers. There, it is imposed that the incumbent firm pays the worker a fixed wage until the next poaching firm appears, and then the two firms bid against each other for the worker. As a result, if the poaching firm is more productive than the incumbent firm, then separation occurs. If the poaching firm is less productive but can nevertheless afford to pay the worker’s current wage, then the incumbent firm would raise the worker’s wage in order to retain him, and this generates within firm wage dynamics.1

Relative to the literature, the strength of our paper is to model explicitly the firm’s responses to the worker’s outside offers, in fact the whole history of his outsider offers, as part of a fully optimal contract. None of the existing works take our approach to insist on fully optimal

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1 As pointed out by Moscarini (2005), this is subject to the following question: if the less productive poaching firm fully anticipates the outcome, why would it bother making the offer in the first place, especially if making offer might impose some costs (no matter how small they are). Our model is not subject to this question by following the traditional search/matching idea where jobs (contracts) are posted, and vacant firms do not engage in any ex post negotiations with the parties they would encounter.
contracting. In particular, in contrast to Postel-Vinay and Robin (2002) and Moscarini (2005) who focus on direct ex post interactions between the incumbent and poaching firms, our strategy is to take the worker’s stochastic outside offers as exogenous and let the firm respond to them according to the ex ante optimally designed contract.  

Our paper is related to the existing studies of optimal dynamic contracting with endogenous termination. Most of the existing models are based on private information, including Spear and Wang (2005), DeMarzo and Fishman (2007), Sannikov (2008), and Wang (2011). In these papers, termination is motivated by the optimal provision of dynamic incentives. Wang and Yang (2011) study a model of optimal dynamic contracting with complete information, where termination is motivated by outside opportunities that the principal-agent relationship is sequentially exposed to. In contrast to our assumption of no commitment from the agent, they assume full commitment. That is, they assume that the agent can fully commit to the terms of any contract he and the principal agree to enter ex ante. The focus of that paper is on how the principal could efficiently utilize the agent’s outside opportunities as an external means for compensating the agent - by way of terminating the contract. The focus of this paper, on the other hand, is on how the contract should allocate self-enforcement optimally over the dynamic path of the contract, and on how termination is used as a device for mitigating the problem of the lack of commitment. The assumption of the lack of commitment is important for the interpretation of our model as a vehicle for studying the dynamics of on-the-job search. The two papers also differ substantially in the results they deliver. In particular, in Wang and Yang (2011), the optimal dynamic contract is a sequence of static contracts where termination occurs with a constant probability, whereas in this paper both the expected utility of the agent and the probability of termination evolve dynamically with the agent’s outside opportunities.

Section 2 lays out the model. Section 3 studies the optimal contract under the assumption that the cost of termination is infinite and so self-enforcement must be implemented in all states of the world. Section 4 studies the optimal contract for the case where the cost of termination is finite. Section 5 concludes the paper. The proofs of lemmas, theorems and corollaries are in the Appendix.

2 Model

Let \( t \) denote time: \( t = 1, 2, \cdots \). There is a single perishable consumption good in the model. There is one principal but a competitive supply of agents. The principal’s objective is to maximize his

\(^2\)Behind our strategy is the idea of the traditional search/matching model where jobs/contracts are publicly posted, and that vacant firms who post the jobs never engage in ex post negotiations with the third parties they would encounter in the market.
expected discounted payoffs. Agents are identical and have the following preferences:

\[ E_\tau \left[ \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t) \right], \]

where \( E_\tau \) denotes the agent’s expectation conditional on information available at the beginning of period \( \tau \), \( \tau \geq 1; \beta \in (0, 1) \) is the discount factor which is shared by the principal and the agent; \( c_t \) and \( u(c_t) \) denote, respectively, the agent’s consumption and utility in period \( t \). Assume \( c_t \in \mathbb{R}_+ \) for all \( t \). That is, consumption must be non-negative.\(^3\) Last, assume the utility function \( u \) is bounded, strictly increasing, strictly concave, twice differentiable, and satisfies the Inada conditions.

Each period, the principal could hire an agent to produce. Each agent, when employed, produces a constant output of \( \theta \geq 0 \). After producing the output, the agent draws a random value \( \xi \) from a set \( \Sigma \) of real numbers. This \( \xi \) represents the value of an outside opportunity for the agent, \( \Sigma \) being the set of all such values. Specifically, the agent’s expected utility would be \( \xi \) if he quits his current job to take this outside offer. Assume

\[ \Sigma = [V_{\text{min}}, V_{\text{max}}] \text{ where } V_{\text{min}} \equiv \frac{u(0)}{1-\beta} \text{ and } V_{\text{max}} \equiv \frac{u(\infty)}{1-\beta}. \]

Here, \( \xi = V_{\text{min}} \) is interpreted as the expected utility of the agent in the state in which he leaves his current job to enter a state of permanent unemployment where his consumption is constantly zero; and \( V_{\text{max}} \) is the agent’s expected utility in the state in which he is entitled to an infinite amount of consumption each period. Last, let \( F: \Sigma \to [0, 1] \) denote the stationary distribution function for the random outside offer \( \xi \) the employed agent receives each period.\(^4\) The agent’s outside offers are perishable - there is no recall of past offers.

The principal and a newly hired agent can enter into a contract that is fully dynamic. Termination is feasible, but costly. There is a fixed cost \( C_0(\geq 0) \) that the principal must incur for terminating and replacing any incumbent agent. Upon termination, the agent leaves the principal to pursue the outside offer he receives and the principal then goes to the external market to hire a new agent. The new agent’s reservation utility is \( V_0 \in \Sigma \).

We make the following assumptions on contracting. First, we assume limited liability on the agent’s part: compensation to the agent must be non-negative. In other words, it is only feasible

\(^3\)As will be clear later, what is important is that the agent’s consumption is bounded from below, but not specifically by zero.

\(^4\)This specification of \( \Sigma \), which imposes that outside offers are bounded from below by \( V_{\text{min}} \), is without loss of generality. Obviously, if an agent is free to receive an outsider offer each period, he must also have the option to quit his current job and then stay permanently out of the labor force. This implies that any outside offer below \( V_{\text{min}} \) need not be considered. Note also that we do not impose any restrictions on the function \( F \). This allows for example the realizations of \( \xi \) to be actually taken from a subset of \( \Sigma \).
that the principal pays the agent (a non-negative amount of compensation), the agent cannot
pay the principal. Second, we assume limited commitment on the agent’s part: each period, the
agent is free to walk away from the contract, before and after receiving his outside offer. This
implies that, if the principal wants to retain an agent who has an outside value of $\xi$, then the
continuation of the contract must promise the agent expected utility of at least $\xi$. The principal,
on the other hand, is able to fully commit. Specifically, the principal can commit to the terms
of the continuation of the contract in any ex post state of the world.

To close this section, as a benchmark for the analysis in the sections to follow, consider the
case where either the agent never receives any outside offers or the agent can fully commit and
the cost of termination is infinite so termination never occurs at the optimum. Then with the
optimal contract the agent’s compensation and expected utility stay constant over time at its
initial level:

$$u(c_t) = (1 - \beta)V, \quad \forall t,$$

where $V$ is the agent’s expected utility at the start of the contract. In other words, the agent’s
consumption is perfectly smooth across current states of the world and over time.

3 The Case $C_0 = \infty$

Suppose it is physically infeasible for the principal to terminate and replace any agent once he
is employed (the cost of termination is infinitely high). As discussed earlier, this assumption is
made implicitly in most of the existing models of dynamic contracting with limited commitment
(e.g., Harris and Holmstrom 1982; Thomas and Worrall, 1988; Phelan, 1995; Kocherlakota, 1996;
Grochulski and Zhang, 2011).

Under the assumption of $C_0 = \infty$, following the tradition of Green (1987) and Spear and
Srivastava (1987), a contract between the principal and the agent, written recursively, takes the
form of

$$\sigma \equiv \{c(V), V_r(\xi; V) : \xi \in \Sigma \text{ and } V \in \Omega\},$$

where $V$ denotes the expected utility of the agent the contract promises to deliver at the beginning
of a period - the “state” variable that summarizes the history of what happened up to that period,
and $\Omega$ is the space for $V$ to take values from - the state space. More precisely, $\Omega$ is the set of
all expected utilities that a feasible contract in this environment (the exact meaning of this to
be specified shortly) can deliver to the agent, at the beginning of any period. Clearly, this state
space is an endogenous variable of the model. Then, for any given $V \in \Omega$, $c(V)$ is the agent’s
consumption in that period, $V_r(\xi; V)$ is the agent’s expected utility next period that the contract
promises to deliver, conditional on the agent’s current outside offer $\xi$, and on $V$.  

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Given the definition of $V$ and the model’s assumptions, $\sigma$ is required to satisfy the following constraints:

$$u(c(V)) + \beta \int_{\Sigma} V_r(\xi; V) dF(\xi) = V, \forall V \in \Omega, \quad (2)$$

$$c(V) \geq 0, \forall V \in \Omega, \quad (3)$$

$$V_r(\xi; V) \geq \xi, \forall \xi \in \Sigma, \forall V \in \Omega, \quad (4)$$

$$V_r(\xi; V) \in \Omega, \forall \xi \in \Sigma, \forall V \in \Omega, \quad (5)$$

where the state space $\Omega$ is the largest set of $V$ that is consistent with the above constraints, or more precisely, $\Omega$ is the largest set of $V$ for which there exists a contract that satisfies (2)-(5), for the same $\Omega$.\(^5\)

In the above, (2) requires that the choice of $c(V)$ and $V_r(\cdot; V)$ be consistent with the definition of $V$. Equation (3) requires compensation be non-negative - the limited liability constraint. Equation (4) is the self-enforcing constraint that requires that, upon the realization of any of the agent’s possible outside offers, the contract promises to give the agent more so he would choose to stay with the principal. Finally, equation (5) says that the utility that the principal promises must be deliverable.

**Lemma 1** $\Omega = [\overline{V}, V_{\text{max}}]$ where $\overline{V}$ solves

$$u(0) + \beta \left( F(\overline{V}) \overline{V} + \int_{\overline{V}}^{V_{\text{max}}} \xi dF(\xi) \right) = \overline{V}. \quad (6)$$

Lemma 1 says that in order for any $V$ to be deliverable, it must be sufficiently high: higher than or equal to $\overline{V}$. Equation (6) gives a Bellman equation for $\overline{V}$, it says that $\overline{V}$ can be delivered in the following way: give the agent zero consumption in the first period, promises him $\overline{V}$ for the next period if his current period outside offer is below $\overline{V}$, match the agent’s outside offer if it is above $\overline{V}$. It is straightforward to show that (6) has a unique solution in the interval $(V_{\text{min}}, V_{\text{max}})$.

Notice quickly that if $V_0 \leq \overline{V}$, then the principal would not need to pay any cost of compensation throughout his life and would then attain a value equal to $\overline{U} \equiv \theta/(1 - \beta)$.

Now for any $V \in \Omega$, let $U^*_{NT}(V)$ denote the maximum value for the principal that can be attained by a contract satisfying (2)-(5). Next, let $B(\Omega)$ denote the space of real-valued functions

\(^5\)This gives $\Omega$ a recursive definition, in a fixed point argument. In the language of Abreu, Pearce, and Stacchetti (1990), $\Omega$ must be the largest self-generating set with respect to (2)-(5). For this, see Wang (1995).
on $\Omega$ that are bounded from above by $\mathcal{U}$, endowed with the topology of pointwise convergence. Then $U_{NT}^* \in B(\Omega)$ must satisfy the following Bellman equation:

$$U_{NT}^* = \Lambda U_{NT}^*,$$  

where $\Lambda : B(\Omega) \rightarrow B(\Omega)$ is a mapping defined as follows: for all $U \in B(\Omega)$ and all $V \in \Omega$,

$$\Lambda U(V) \equiv \max_{c,V_r(\xi) : \xi \in \Sigma} \left\{ (\theta - c) + \beta \int_\Sigma U(V_r(\xi))dF(\xi) \right\}$$

subject to

$$u(c) + \beta \int_\Sigma V_r(\xi)dF(\xi) = V,$$  

$$c \geq 0,$$  

$$V_r(\xi) \geq \xi, \forall \xi \in \Sigma,$$  

$$V_r(\xi) \in \Omega, \forall \xi \in \Sigma.$$  

**Theorem 1** (i) The principal’s value function $U_{NT}^*$ is strictly decreasing, strictly concave, and can be attained asymptotically by applying the mapping $\Lambda$ repeatedly on $U$. (ii) The optimal contract has: for all $V \in \Omega$,

$$V_r(\xi, V) = \max\{\xi, V\}, \forall \xi.$$  

The key part of the proof of Theorem 1 is to show that the value function $U_{NT}^*$ is concave, which is achieved by constructing a sequence of concave functions that converges to it. Theorem 1 says that, with the optimal contract, either the principal ignores the agent’s outside offer - when it is below his current promised utility, or the principal matches the agent’s outside offer to retain him - when it exceeds his current promised utility. The logic behind this is simple. (a) If the self-enforcing constraint (10) is not binding, then $V_r(\xi) = V$ - the first best intertemporal smoothing at $(V, \xi)$ is attained. (b) If (10) is binding, then of course $V_r(\xi) = \xi$. (c) (10) binds if and only if $\xi \geq V$.

Figure 1 depicts the optimal contract in a two dimensional graph where the horizontal axis represents $V$ and the vertical axis $\xi$. By Theorem 1, in any state $(V, \xi)$ in the area below the 45 degree line (and with $V \geq \overline{V}$), the self-enforcing constraint is not binding and the agent is
given a promised utility equal to his current utility $V$; in the states above the 45 degree line, the self-enforcing constraint is binding and the optimal contract specifies $V_r(\xi) = \xi$.

Figure 1 gives an obvious sense that the commitment problem diminishes in the agent’s expected utility $V$, and increases in his current outside offer $\xi$. Specifically, for any given $\xi$, the self-enforcing constraint is binding if and only if $V$ is sufficiently low (below $\xi$); and for any given $V$, the self-enforcing constraint is binding if and only if $\xi$ is sufficiently high (above $V$). This is intuitive. A higher $\xi$ requires a higher promised utility to be given to the agent in order to induce him to stay (it increases the value of the right hand side of the constraint (10)). A higher $V$ dictates higher utilities be promised to the agent in all states of the agent’s outside offer $\xi$, relaxing the self-enforcing constraint.

![Figure 1: The Optimal Contract with $C_0 = \infty$](image)

Given the above, the dynamics the optimal contract generates should over time push the agent up, from lower to higher $V$s, so as to reduce the problem of the lack of commitment monotonically.
over time. In fact this is what the model predicts. Over time, higher and higher outside offers are matched to push the agent’s expected utility up monotonically to higher and higher levels. Meanwhile, over time, the probability the principal matches the agent’s outside offer gets lower and lower and eventually converges to zero as the problem of the lack of commitment becomes less and less severe and eventually diminishes to null.

**Corollary 1** The agent’s compensation is back-loaded with the optimal contract: for all \( V \),

\[ u(c(V)) < (1 - \beta)V. \]

The agent’s compensation is backloaded because his utility is backloaded. His utility is back-loaded because over time the self-enforcing constraint forces the principal to match higher and higher outside offers of the agent so that he has the incentives to stay. Back-loading as a means for providing incentives is an old idea in economics. In two of the most related examples, Edwards (1981) and Burdett and Coles (2003), back-loading in compensation builds a collateral against worker quitting. So far in the literature though, back-loading has not been modeled as an outcome of the principal’s optimal response to the agent’s outside offers. In Burdett and Coles (2003) for example, wage increases optimally with the worker’s tenure but is not directly contingent on his outside offers.

### 4 The Case \( C_0 < \infty \)

When the cost of termination is finite, termination maybe optimal. There are three motives for termination in our model. (a) Any outside offer represents an opportunity the principal could use as an external means for fulfilling his promise to the agent, and termination of the current relationship makes this possible. (b) Terminating the current relationship may allow the principal to start a new relationship with a less expensive agent. (c) Termination can be used as a means for mitigating the problem of the lack of commitment - as discussed in the introduction of the paper. In this section, we study how these motives for termination work jointly to determine the dynamics of the optimal contract.

Use the agent’s beginning-of-period expected utility, denoted \( V \), as a state variable, a dynamic contract, defined recursively, is

\[ \sigma \equiv \{c(V), I(\xi; V), V_r(\xi; V) : \xi \in \Sigma \text{ and } V \in \Phi\}, \]

where \( c(V) \) is the agent’s compensation in the current period; \( I(\xi; V) \) is the probability with which the agent is retained conditional on his current outside offer being \( \xi \); \( V_r(\xi; V) \) is the agent’s expected utility next period if his current outside offer is \( \xi \) and he is retained; and finally,
Φ is the space for V - the set of all expected utilities that the contract can deliver - which is also an endogenous variable of the model.

A contract σ is feasible if it satisfies the following constraints:

\[ u(c(V)) + \beta \int_{\Sigma} [I(\xi; V) V_r(\xi; V) + (1 - I(\xi; V)) \xi] dF(\xi) = V, \forall V \in \Phi, \]  
(13)

\[ c(V) \geq 0, \forall V \in \Phi, \]  
(14)

\[ I(\xi; V)(1 - I(\xi; V)) \geq 0, \forall \xi \in \Sigma, \forall V \in \Phi, \]  
(15)

\[ V_r(\xi; V) \geq \xi, \forall \xi \in \Sigma, \forall V \in \Phi, \]  
(16)

\[ V_r(\xi; V) \in \Phi, \forall \xi \in \Sigma, \forall V \in \Phi, \]  
(17)

where Φ is the largest set of the V’s that is consistent with the above constraints (the largest self-generating set with respect to the constraints).

In the above, equation (13) is the promise-keeping constraint that requires that the choices of the current variables be consistent with the definition of V; Equation (14) is the limited liability constraint; Equation (15) holds if and only if \( I(\xi; V) \in [0, 1] \), for all \( \xi \in \Sigma \) and \( V \in \Phi \).\(^6\) Equation (16) is the self-enforcing constraint which says that if the principal retains the agent, then the agent must be given no less than \( \xi \). Equation (17) requires that if the agent stays, then he must be given a promised utility that the contract can deliver.

Notice that the self-enforcing constraint (16) and the deliverability constraint (17) are imposed only in the states of retention. This immediately gives a sense that (weakly) less self-enforcement are imposed if the contract prescribes more termination. Remember in the case of \( C_0 = \infty \), since termination is not feasible, (16) and (17) must imposed in all states of the world.

**Lemma 2** \( \Phi = [u(0) + \beta E(\xi), V_{\text{max}}] \).

So, relative to the case where the cost of termination is infinite, termination expands the set of expected utilities deliverable, allowing the contract to deliver any level of expected utility

\(^6\)To formulate \( I(\xi; V) \in [0, 1] \) as (15) has a technical advantage: it reduces the number of Lagrangian multipliers by one in the analysis to follow.

\(^7\)Note that randomized termination is allowed under (15). In the case of deterministic termination where \( I(\xi) \) must be either 0 or 1, we need only replace (15) with \( I(\xi)(1 - I(\xi)) = 0, \forall \xi \in \Sigma. \)
below \( V \) but above \( u(0) + \beta E(\xi) \). To understand this, compare what happens in the states of termination and retention. Suppose termination occurs at an arbitrary \( \xi \). Then the agent receives \( \xi \), leaves, his relationship with the current principal ends. Suppose the agent is retained. Then he must be given an expected utility that’s not only better than \( \xi \) but also consistent with the fact that retention gives him opportunities to receive more outside offers which he can potentially use for higher utilities. This immediately implies that the contract must promise weakly more utilities to the agent in the state of retention than termination. And this should especially be so when \( \xi \) is low. In fact, as we will see, this very logic could dictate that the optimal contract prescribes termination in a lower rather than a higher state of \( \xi \) when the problem of the lack of commitment is sufficiently severe.\(^8\)

For all \( V \in [u(0) + \beta E(\xi), V_{\text{max}}] \), let \( U^*(V) \) denote the principal’s value conditional on \( V \). Call \( U^*: [u(0) + \beta E(\xi), V_{\text{max}}] \to \mathbb{R} \) the principal’s value function. This function must satisfy the following Bellman equation:

\[
U^* = \Gamma U^*,
\]

where \( \Gamma \) is defined as follows: for all \( U \in B(\Phi) \) and all \( V \in \Phi \),

\[
\Gamma U(V) \equiv \max_{c, I(\xi), V_r(\xi) : \xi \in \Sigma} \left\{ (\theta - c) + \beta \int_{\Sigma} [I(\xi)U(V_r(\xi)) + (1 - I(\xi))(U_0(U) - C_0)]dF(\xi) \right\}
\]  

subject to

\[
u(c) + \beta \int_{\Sigma} [I(\xi)V_r(\xi) + (1 - I(\xi))\xi]dF(\xi) = V,
\]  

\[
c \geq 0,
\]  

\[
I(\xi)(1 - I(\xi)) \geq 0, \forall \xi \in \Sigma,
\]  

\[
V_r(\xi) \geq \xi, \forall \xi \in \Sigma,
\]  

\[
V_r(\xi) \in \Phi, \forall \xi \in \Sigma,
\]

where

\[
U_0(U) \equiv \max_{V \in \Phi} \{U(V)\} \text{ s.t. } \tilde{V} \geq V_0.
\]

\(^8\)Note that termination allows a lower promised utility to be delivered not because it allows the contract to avoid matching the more expensive outside offers.
Above, (22) is the self-enforcing constraint, (23) is the deliverability constraint requiring that the agent’s promised utility be deliverable. Equation (24) says that after the termination the principal would go back to the external market to hire a new agent who will be given an expected utility \( \bar{V} \) that maximizes the principal’s value, subject to the condition that this expected utility is above the agent’s reservation \( V_0 \) and is deliverable with a feasible contract \( (\bar{V} \in \Phi) \).

**Theorem 2**

(i) The principal’s value function \( U^* \) is concave, attains its maximum at \( V = \bar{V} \), and can be attained asymptotically by applying the mapping \( \Gamma \) repeatedly on \( U \).

(ii) The following contract is optimal: for all \( V \in \Phi \), there exist \( \bar{\xi}_1 \in [V_{\min}, \bar{V}] \) and \( \bar{\xi}_2 \in [\bar{V}, V_{\max}] \) (\( \bar{\xi}_1 \) and \( \bar{\xi}_2 \) depending on \( V \)) such that

\[
I(\xi) = \begin{cases} 
1 & \text{for } \xi \in (\bar{\xi}_1, \bar{\xi}_2) \\
0 & \text{for } \xi < \bar{\xi}_1 \text{ and } \xi > \bar{\xi}_2 \end{cases};
\]

\[ V_r(\xi) = \max\{\xi, V\}, \forall \xi. \]

(iii) \( \bar{\xi}_1 > V_{\min} \) for all \( V < \bar{V} \). Moreover, \( \bar{\xi}_1 \) is strictly decreasing in \( V \) over the interval \([u(0) + \beta E(\xi), \bar{V}]\); and \( \bar{\xi}_1 = V_{\min} \) for all \( V \geq \bar{V} \).

(iv) There exists \( V'' \geq \bar{V} \) with \( U^*(V'') = U_0(U^*) - C_0 \) such that \( \bar{\xi}_2 = V'' \) for all \( V \leq V'' \) and \( \bar{\xi}_2 \) is strictly increasing in \( V \) for \( V > V'' \).

(v) With the optimal contract, the principal matches the agent’s outside offer to retain him (the self-enforcing constraint is binding) if and only if \( V \leq \xi \leq V'' \).

Thus, with the optimal contract, except at the cutoffs \( \bar{\xi}_1 \) and \( \bar{\xi}_2 \), termination is deterministic, not stochastic. And the optimal contract, by Theorem 2, is fully characterized by these two cutoffs, where \( \bar{\xi}_1 \) is the cutoff level of the agent’s outside offer below which he is terminated, \( \bar{\xi}_2 \) is the cutoff level of his outside offer above which he is terminated. Figure 2 depicts these cutoffs, each as a function of \( V \). The principal’s value function \( U^* \) is depicted in Figure 3. The principal’s value \( U^*(V) \) attains its maximum value at \( V = \bar{V} \). Remember the principal achieves its maximum value at the same \( \bar{V} \) in the case of infinite termination cost.

Theorem 2 states that for all \( V \), termination occurs if and only if the agent’s outside offer is sufficiently good (above \( \bar{\xi}_2 \)) or sufficiently bad (below \( \bar{\xi}_1 \)). A larger \( \xi \) makes termination more worthwhile because accepting it allows the agent to catch a better value which, in turn, implies the principal would save more on the cost of compensating the agent. A larger \( \xi \) also makes self-enforcement more costly: in order to induce the agent to stay the principal must now give him more, and this makes termination a more attractive alternative than continuation. These two effects work in the same direction to explain why sufficiently high outside offers induce
retention, and this makes self-enforcement in the case of limited commitment less likely to be the case of full commitment would dictate lower promised utilities for the agent in the states of since a lower \( V \) may not guarantee retention, noticing, specifically, that for any \( V < \bar{V}, \xi_1 > V_{\min} \), and thus any \( \xi \in [V_{\min}, \xi_1) \) would trigger termination. If a small \( \xi \) does not offer much benefit for the principal to use it as an external means of compensation, why terminating at it? We will answer this question later, but notice first that this type of termination would not occur on the equilibrium path. This is because it is never optimal to start a new worker with expected utility below \( \bar{V} \), and once started, the worker’s expected utility will increase monotonically over time until he is terminated.

Observe next that \( \bar{\xi}_2 \) is increasing in the agent’s promised utility \( V \). Put differently, termination that occurs because the agent’s outside offer is sufficiently high diminishes as \( V \) increases. Since a lower \( V \) is associated with a more severe commitment problem (remember a lower \( V \) in the case of full commitment would dictate lower promised utilities for the agent in the states of retention, and this makes self-enforcement in the case of limited commitment less likely to be
consistent with the first best allocation, or the self-enforcing constraint is more likely to be binding), there is higher demand for utilizing termination to avoid a binding self-enforcing constraint, termination then occurs with a higher probability at a lower rather than a higher $V$.

As in the case of $C_0 = \infty$, if the agent is retained in a state $(\xi, V)$ with $\xi \leq V$, then the self-enforcing constraint is not binding, first best intertemporal allocation is attained in this state with $V_r(\xi; V) = V$. That is, if the agent is retained upon an outside offer inferior to his current promised utility $V$, then his promised utility next period will continue to be $V$. On the other hand, if the agent is retained at a $\xi > V$, the full-commitment optimality condition $V_r(\xi) = V$ must be violated in order to provide incentives for the agent to stay and, as a result, the self-enforcing constraint is binding and the optimal contract dictates that the principal match the agent’s outside offer to induce him to stay.

Thus, with the optimal contract, the self-enforcing constraint is imposed and binding in a specific state of the world if and only if the contract finds it optimal for the principal to match the agent’s outside offer in that state. This, in turn, occurs if and only if (i) termination is not optimal and (ii) the agent’s outside offer exceeds his current promised utility. Obviously, (ii) occurs with a lower probability for a higher $V$. Figure 2 shows that (i) and (ii) never occur simultaneously for $V_s$ sufficiently high: higher than $V''$. Put differently, for a “richer” agent (with a higher $V$), his outside offers are less likely to be matched; and the optimal contract never seeks to match the agent’s any outside offer if he is sufficiently rich.

Observe that the principal never offers to match the agent’s outside offer beyond the constant upper bound $V'': \bar{\xi}_2$ is flat in $V$ for $V \leq V''$. This is interesting. This implies that the agent’s expected utility on the path of retention is capped by $V''$, provided that he starts the contract below $V''$. To explain this, for any $V$, let $\bar{\xi}$ denote the cutoff at which the contract is indifferent between termination and retention (retention achieved by matching the agent’s outside offer). Then it must hold that, at the margin between termination and retention, the net gains from the act of switching from retention to termination be zero at $\bar{\xi}$:

$$\begin{align*}
(U_0(U^*) - C_0) - U^*(\bar{\xi}) &= 0,
\end{align*}$$

(27)

where remember $U_0(U^*)$ is the principal’s value with the new agent hired to replace the existing one. Note that at $\bar{\xi}$, the agent is indifferent between being terminated and being retained: he receives the same $\bar{\xi}$, terminated or retained. For the principal, switching from retention to termination gives him a net benefit that consists of two components: one measures the net benefit from switching to a new agent $(U_0(U^*) - U^*(\bar{\xi}))$, the other the fixed cost $C_0$ of termination.

Obviously, the solution of $\bar{\xi}$ to (27) is constant in $V$ and it is just our $V''$. Notice that $V''$ is the critical level of $V$ below which the self-enforcing constraint is binding with a positive probability
(with respect to the random outside offer $\xi$) and beyond which the self-enforcing constraint is not binding at all. This gives us a sense that $V''$ is a measure for how serious the problem of the lack of commitment is.

Over the interval $[u(0) + \beta E(\xi), \bar{V}]$, the principal’s value function $U^*$ is increasing. Over the same interval, termination occurs not only for $\xi$ sufficiently high (above $V''$) but also for $\xi$ sufficiently low (below $\bar{\xi}_1 (\geq V_{\min})$). These features of the optimal contract are connected, and we explain how. Imagine lowering the value of $V$ from $\bar{V}$ to its minimum level deliverable with a feasible contract, i.e., $u(0) + \beta E(\xi)$. From promise-keeping, a lower $V$ forces the contract to reduce the values of $u(c)$ and $V_r(\xi)$. But, for $V$ below $\bar{V}$, there are two factors that limit the contract’s ability to do so:

One. For all $\xi \geq V$, limited commitment dictates that the value of $V_r(\xi)$ must be at least $\xi$, whether the agent is retained, or terminated.

Two. For all $\xi \leq V$ where the agent is retained, optimal risk sharing requires $V_r(\xi) = V$. That is, for these $\xi$, optimal risk sharing requires that the agent be paid more than his outside offers.

It then follows that, for any $V$ sufficiently close to $u(0) + \beta E(\xi)$, termination must occur for at least some $\xi \in [V_{\min}, V]$.\(^9\) Given this, it is optimal to let termination occur at a lower, rather than a higher $\xi$. Why? There is a trade off. On the one hand, when the agent is retained at a lower $\xi$, a greater amount of extra value (a larger $V - \xi$) must be given him. That is, including a lower $\xi$ in the retention region makes optimal risk sharing more costly. On the other hand, terminating the agent at a low $\xi$ (a $\xi$ that is near $V_{\min}$) brings the principal less benefits, for the same fixed cost of termination. Remember termination allows the $\xi$ to be used as an external means for compensating the agent. Theorem 2 says that the first effect dominates.

A less formal but more straightforward way to explain why termination occurs at a lower rather than a higher $\xi$ is to divide the argument into the following two steps.

Step 1. Given that a lower $V$, by way of tightening up the self-enforcing constraint, imposes more termination on the contract, the principal’s value is lower, not higher, for lower values of $V$. This explains why the firm’s value function is increasing over the interval $[u(0) + \beta E(\xi), \bar{V}]$.

\(^9\)In particular, at $V = u(0) + \beta E(\xi)$, if there is no termination over the interval $[V_{\min}, V]$. Then the minimum expected utility the contract can deliver is

$$u(0) + \beta \left( F(u(0) + \beta E(\xi))V + \int_{u(0) + \beta E(\xi)}^{V_{\max}} \xi dF(\xi) \right) > u(0) + \beta E(\xi).$$
Step 2. Terminating the agent at a lower $\xi$ gives the agent a lower utility in the state of termination which allows the principal to promise the agent a higher utility in the state of retention (the promise-keeping constraint) which, in turn, because of the upward sloping value function, gives the principal a higher value. This provides the motivation for terminating the agent after he receives a low outside offer.\footnote{As noted earlier, this type of termination never occurs on the equilibrium path. As such, this certainly is not a result of the paper we wish to advocate. Nevertheless, the economic intuition behind it is in some ways novel and interesting. Question is, would the economic logic behind this result be able to generate more meaningful outcomes in an environment that is richer than our current? We speculate that the answer is “yes”. For example, imagine adding moral hazard to our current model of complete information. The optimal provision of intertemporal incentives may then generate dynamics for the worker’s expected utility to decrease over time, eventually generating a state in which the contract may be terminated after a lower outside offer is received, for the same reason we have just discussed. We leave this interesting extension for future research.}

The remainder of this section highlights the paper’s main ideas and results, as well as the economic logic behind them.

### 4.1 Termination as a Device for Mitigating the Problem of the Lack of Commitment

In any given state of the world in the model, if termination occurs, then the contract need not enforce continuation in that state, then it does not matter whether the agent can commit to staying with the principal in that state, and it is through this termination helps alleviate the problem of the lack of commitment. We now summarize several aspects of the optimal contract that illustrate this role termination plays.

(i) Termination, because it helps alleviate the problem of the lack of commitment, expands the set of expected utilities the contract can deliver (Lemma 2 and the remark following it).

(ii) For any deliverable $V$, termination reduces the set of the agent’s outside offers at which a binding self-enforcing constraint is imposed with the optimal contract. Compare Figures 1 and 2. In particular, termination reduces the problem of the lack of commitment to zero - no self-enforcement is needed - for all $V$ above $V''$.

(iii) The ability with which termination can be used as a device for mitigating limited commitment depends on the cost of termination. When the cost of termination is lower, termination is used more and the optimal contract then need only impose self-enforcement over a smaller set of the agent’s outside offers. Observe from equation (27) that if $C_0 = 0$, then $V'' = V$. In general, equation (27) indicates that $V''$ is increasing in $C_0$. When $C_0 \to \infty$, $V'' \to V_{\text{max}}$ and the optimal contract then converges to that for $C_0 = \infty$ where the commitment constraint is binding for all $\xi \geq V$ for all $V$.
4.2 Dynamics and Convergence

The model generates dynamics that the literature has not been able to establish. Starting initially from any $V < V''$ (so the lack of commitment is a binding constraint), so long as the relationship continues, it moves up monotonically in expected utility and, over time, converges to, but never actually attains, $V''$. Over time, the agent’s promised utility increases each time he receives an outside offer that exceeds his current promised utility and the principal matches that offer. Along the path of the optimal contract, the principal never offers any voluntary raises to the agent.

In the limit at $V''$, the continuation of the optimal contract is such that it is composed of a sequence of static contracts each of which offers retention at the same promised utility $V''$ but never responds to any of the agent’s outside offers. Thus, in the limit, the agent either stays voluntarily with the principal and continues to receive the constant promised utility, or he quits voluntarily to pursue a better outside opportunity. This limiting contract is exactly Burdett (1978).

That the convergence occurs on the path of retention is because the optimal contract generates a monotonic and bounded sequence of promised utilities. That the sequence is monotonic is because the agent’s promised utility moves only when his outside offer exceeds his current promised utility and the principal acts to match that offer, and this results from the interaction between the nature of the commitment constraint and the desire for the optimal contract to allocate utility evenly over time. That the dynamics is bounded from above and away from $V_{\text{max}}$ is because of the use of termination in the optimal contract. Since sufficiently high outside offers (higher than $V''$ specifically) always trigger termination, the agent, upon retention, never would be given a level of promised utility that exceeds $V''$.

4.3 Stationarity and Involuntary Terminations

The model generates also optimal contracts that are completely stationary, depending on the starting expected utility of the agent. Specifically, for any current expected utility of the agent $V$ with $V > V''$, the principal never matches his outside offer and, conditional on retention, his expected utility next period remains constant at $V$.$^{11}$ The optimal contract has a much simpler structure in this case. For any given level of the agent’s expected utility, it is composed of a sequence of identical static contracts and the principal-agent relationship ends stochastically according to a simple rule: there is a constant critical level of the agents outside offer below which the ongoing relationship continues and above which the parties separate.

$^{11}$Theorem 2, equation (26), and notice that $\bar{V}_2 < V$. 19
Thus for $V$ sufficiently large the optimal contract in our model resembles the contracts in the classic models of on-the-job search (Burdett, 1978; Burdett and Coles, 2003). The difference is, all terminations are voluntary in the existing on-the-job search models whereas our model generates both voluntary and involuntary terminations. Specifically, for any $\xi \geq V$, termination is voluntary - the agent is (weakly) better off leaving the principal than staying with him. For any $\xi \in (\bar{\xi}_2, V)$, termination is involuntary - the agent is better off continuing on with the current relationship than leaving it.

Notice that the cutoff for termination, $\bar{\xi}_2$, being strictly less than $V$, the worker’s current period expected utility, is important for the stationarity of the optimal contract and the involuntary termination that arises with it. Notice also that this is not the case for $V$ sufficiently low. For any $V < V''$, the optimal contract is not stationary and all terminations are voluntary. What happens is that a sufficiently high expected utility of the agent, by making him sufficiently expensive for the principal, provides a sufficiently strong motivation for termination. As such, it is never optimal for the principal to match the agent’s outside offer, and outside offers that provide relatively lower values for the agent - lower than the value his current contract offers - are accepted.

4.4 Back-loading in Compensation

Does the optimal contract still show back-loading in the agent’s utility over time? Under a specified condition, the answer is “yes”.

**Corollary 2** Suppose

$$-\frac{u''(c)c}{u'(c)} \geq \frac{1}{1 - \beta}, \forall c \geq u^{-1}((1 - \beta)V).$$

Then the optimal contract has $u(c(V)) < (1 - \beta)V$ for all $V \in \Phi$.

Remember if the agent were not to receive any outside offers in the model, then his compensation and utility would be loaded evenly over time. Here, there are two reasons why the optimal contract is back-loaded. First, backloading is an outcome of the principal’s optimal response to the agent’s outside offers, conditional on retention, as in the case of $C_0 = \infty$ in Section 3. Second, with a back-loaded contract, compensation is delayed, maybe avoided. A back-loaded compensation scheme allows the contract to shift the burden of compensation from his current principal to the future principal, increasing the current principal’s value.
4.5 Outside Opportunities: Good or Bad?

As discussed in the introduction, existing models of limited commitment assume that termination
is not feasible and impose a self-enforcing constraint in all states of history. In these models
then, the agent’s outside opportunities are always bad from the principal’s perspective: they
could never be used but they do impose a cost on the contract. In contrast to the implications
of these models, our paper illustrates a sense that, from the principal’s perspective, the outside
offers that the agent is exposed to may not be a bad thing - they can be either good or bad. A
specific outside offer can be good because it offers an opportunity for the principal to use it as
an external means of fulfilling the principal’s promises to the agent. It can also be bad because
it may give the agent leverage against the principal, forcing the principal to match his outside
offer.

Our model offers a clear characterization for when the outside offers are good and when they
are bad. To obtain this characterization, compare the principal’s value function in this section
(random outside offers, limited commitment, costly termination), \( U^* \), with that in an assumed
scenario in which the agent is not exposed to any outside offers (and so all compensation to
the agent - which is constant across period as in equation (1) - must come internally from
the principal’s pocket), \( \hat{U} \). Figure 3 illustrates this comparison. It shows that the outside
opportunities are good for the principal when \( V \) is sufficiently high (above \( V' \)), bad when the \( V \)
is sufficiently low (below \( V' \)).

The intuition for this result should now be clear. A higher \( V \) allows the principal to gain
in the case with outside offers than with not in two ways. First, a higher \( V \), by relaxing the
self-enforcing constraint, allows the contract to achieve more intertemporal utility smoothing (in
particular, first-best intertemporal allocation is achieved in more states of the agent’s outside
offer). Second, a higher \( V \), since it is associated with a less severe commitment problem, reduces
the burden on termination as a device for tackling no commitment, resulting in the more efficient
use of the agent’s outside offers as an external means of compensation. For this, notice \( \xi_2 \) increases
in \( V \) for \( V \geq V'' \). With a higher \( V \), less terminations occur, higher outside offers are taken.

5 Conclusion

In this paper, we have constructed and studied a model in which an ongoing principal-agent
relationship is subject to a sequence of stochastic outside opportunities for the agent. The agent
cannot commit to staying with the principal - he is free to pursue any outside offer he receives.
Termination allows the principal to utilize the agent’s outside offers as an external means for
providing compensation to the agent. Termination is also used as a device for alleviating the
problem of the lack of commitment. Optimality is achieved through simultaneously determining where to enforce continuation in the space of the agent’s history of outside offers, and where to call for termination of the relationship. It turns out that optimal self-enforcement and termination generate interesting dynamics that provides a theoretical justification for the contract used in Burdett (1978)’s classic work of on-the-job search.

In the real world, many long-term economic relationships, like the one we have studied here, are subject to external shocks that generate temptations for one or more parties of the relationship to leave to pursue the values of an outside opportunity. Our paper presents one theoretical attempt to study the dynamics of these relationships. Obviously, many extensions to our model are possible as interesting future research topics, including taking the self-enforcement/termination issue to a general equilibrium framework.
Appendix

A Proof of Lemma 1

Step 1 We show that if $V \in \Omega$, then $V \geq \bar{V}$. Let $\bar{V} \equiv \min\{\Omega\}$. It suffices to show $\bar{V} \geq \nabla$. Given $\bar{V} \in \Omega$, there exists a feasible contract $\sigma = \{c(V), V_r(\xi; V): \xi \in \Sigma \text{ and } V \in \Omega\}$ satisfying (2)-(5) such that

$$\bar{V} = u(c(\bar{V})) + \beta\int_{\Sigma} V_r(\xi; \bar{V})dF(\xi) \geq u(0) + \beta\int_{\Sigma} \max\{\xi, \bar{V}\}dF(\xi) = u(0) + \beta\left(F(\bar{V})\bar{V} + \int_{\bar{V}}^{V_{\max}} \xi dF(\xi)\right),$$

where the first equality follows from (2), and the inequality follows from $c(\bar{V}) \geq 0$ by (3) and $V_r(\xi; \bar{V}) \geq \max\{\xi, \bar{V}\}$ for all $\xi \in \Sigma$ by (4), (5), and $\bar{V} \equiv \min\{\Omega\}$. Thus,

$$Y(\bar{V}) = \bar{V} - \left[u(0) + \beta\left(F(\bar{V})\bar{V} + \int_{\bar{V}}^{V_{\max}} \xi dF(\xi)\right)\right] \geq 0.$$

Given $Y'(\bar{V}) = 1 - \beta F(\bar{V}) > 0$ and $Y(\bar{V}) = 0$ by (6), it must then hold that $\bar{V} \geq \nabla$.

Step 2 We show that for any $V \in [\bar{V}, V_{\max})$, there exists a feasible contract that attains it. Consider contract $\sigma = \{c(V'), V_r(\xi, V'): \xi \in \Sigma, V' \in [\bar{V}, V_{\max})\}$ where, for each $\xi$ and $V'$,

$$V_r(\xi; V') = \max\{\xi, V'\},$$

and $c(V')$ is chosen to be such that $c(V') \geq 0$ and

$$u(c(V')) + \int_{\Sigma} V_r(\xi, V')d\xi = V'.$$

It is straightforward to show that such $c(V')$ exists for all $V' \in [\bar{V}, V_{\max})$. As such, $\sigma$ is feasible and attains any $V' \in [\bar{V}, V_{\max})$, including $\bar{V}$. This proves the lemma.

B Proof of Theorem 1

By Lemma 1, constraints (10) and (11) can be combined to read

$$V_r(\xi) \geq \max\{\xi, \bar{V}\}, \forall \xi \in \Sigma.$$  \hspace{1cm} (28)

The first prove (i) of the theorem. This takes five steps.
Step 1 We show that if $U \in B(\Omega)$ is decreasing, then $\Lambda U$ is decreasing. Let $U \in B(\Omega)$ be decreasing. Let $V^1, V^2 \in \Omega$ with $V^1 < V^2$. Suppose \{c^2, V^2_r(\xi) : \forall \xi \in \Sigma\} is optimal at $V = V^2$. Let $k \equiv \frac{V^1 - V}{V^2 - V} \in (0, 1],$
\[
c^1 = u^{-1}((1 - k)u(0) + ku(c^2)) \leq c^2,
\]
$V^1_r(\xi) = (1 - k) \max \{\xi, V\} + kV^2_r(\xi) \leq V^2_r(\xi), \forall \xi \in \Sigma.$

Then it is straightforward to show that \{c^1, V^1_r(\cdot)\} satisfy (8), (9), and (28) at $V = V^1$ and that
\[
\Lambda U(V^1) \geq (\theta - u^{-1}((1 - k)u(0) + ku(c^2))) + \beta \int_\Sigma U((1 - k)\max \{\xi, V\} + kV^2_r(\xi))dF(\xi)
\]
\[
\geq (\theta - c^2) + \beta \int_\Sigma U(V^2_r(\xi))dF(\xi)
\]
\[
= \Lambda U(V^2),
\]
where the second inequality follows from the assumption that $U$ is decreasing. So $\Lambda U$ is decreasing.

Step 2 We show that if $U \in B(\Omega)$ is concave, then $\Lambda U$ is concave. Suppose $U$ is concave. For any given $V^1, V^2 \in \Omega$, suppose \{c^i, V^i_r(\xi) : \forall \xi \in \Sigma\} is optimal at $V = V^i$, $i = 1, 2$. Then it is straightforward to show that \{c, V_r(\xi), \xi \in \Sigma\}, where
\[
c = u^{-1}(ku(c^1) + (1 - k)u(c^2)),
\]
$V_r(\xi) = kV^1_r(\xi) + (1 - k)V^2_r(\xi), \forall \xi \in \Sigma,$

satisfy (8), (9), and (28) at $V = kV^1 + (1 - k)V^2$ for all $k \in [0, 1]$. Moreover,
\[
\Lambda U(V) \geq (\theta - u^{-1}(ku(c^1) + (1 - k)u(c^2))) + \beta \int_\Sigma U(kV^1_r(\xi) + (1 - k)V^2_r(\xi))dF(\xi)
\]
\[
\geq \beta \int_\Sigma \{[\theta - (kc^1 + (1 - k)c^2)] + \beta \int_\Sigma [kU(V^1_r(\xi)) + (1 - k)U(V^2_r(\xi))]dF(\xi)\}
\]
\[
= k\Lambda U(V^1) + (1 - k)\Lambda U(V^2),
\]
where the second inequality holds because both $-u^{-1}$ and $U$ are concave.

Step 3 We show that \{\Lambda^n U\}_{n=0}^\infty \subseteq B(\Omega)$ is a sequence of decreasing and concave functions converging monotonically to $U^*_NT$, which is then decreasing and concave.

First, we show that $U^*_NT$ is the maximum fixed point of $\Lambda$ in the following sense: there does not exist any fixed point of $\Lambda$ denoted $\tilde{U}$ such that $\tilde{U}(V) > U^*_NT(V)$ for some $V$. Suppose otherwise. Fix any $V$ with $\tilde{U}(V) > U^*_NT(V)$. A contract can be constructed recursively to
deliver the agent expected utility $V$ and the principal expected payoff $\tilde{U}(V) > U_{NT}^*(V)$, which contradicts with the definition of $U_{NT}^*(V)$.

Second, we show $U_{NT}^*(V) \leq \Gamma^{n+1} \bar{U}(V) \leq \Gamma^n \bar{U}(V) \equiv \bar{U}(V) \equiv \theta/(1 - \beta)$ for all $V$ and $n = 1, 2, \cdots$. It is straightforward to show $U_{NT}^*(V) \leq \bar{U}(V) \leq \bar{U}(V)$ for all $V$. Given that $\Lambda$ is monotonic, the desired result then follows.

Third, we show that the function $U_{NT}^*$ is decreasing and concave. Since $\Gamma\bar{U}$ is a fixed point of $\Gamma$ with $\Gamma\bar{U}(V) \geq U_{NT}^*(V)$ for all $V$, by Step 1 we must have $U_{NT}^* = \Gamma\bar{U}$. Moreover, given $\bar{U}$ is decreasing and concave, $\Gamma^n\bar{U}$ is decreasing and concave for all $n$, by Steps 1 and 2. Hence, $\Gamma\bar{U}$ or $U_{NT}^*$ is decreasing and concave.

**Step 4** We show that $U_{NT}^*$ is differentiable. The proof applies Theorem 4.10 of Stokey and Lucas (1989). Basically, given that $U_{NT}^*$ is concave, to show it is differentiable at any given $\tilde{V}$, we need only construct a concave function $W(V)$ which takes as its domain a neighborhood of $\tilde{V}$, is uniformly below $U_{NT}^*$ but attains the same value as does $U_{NT}^*$ at $\tilde{V}$. In the following, we ignore the case $\tilde{V} = \bar{V}$ and take as given that $\tilde{V} > \bar{V}$.

Let $c^*(\tilde{V})$ be the optimal $c$ at $\tilde{V}$.

**Case 1** $c^*(\tilde{V}) > 0$. The function $W$ is constructed as: for all $V$,

$$W(V) \equiv (\theta - u^{-1}(u(c^*(\tilde{V}))) + (V - \tilde{V})) + \beta \int_{\Sigma} U_{NT}^*(V_r(\xi; \tilde{V}))dF(\xi).$$

This function is concave and differentiable, given that $u$ is concave and differentiable. Moreover, $W(V) \leq U_{NT}^*(V)$ and $W(\tilde{V}) = U_{NT}^*(\tilde{V})$, given that $c \equiv u^{-1}(u(c^*(\tilde{V}))) + (V - \tilde{V})$ and $V_r(\xi; \tilde{V})$ for all $\xi$ satisfy (8)-(11) at $\tilde{V}$. Thus $U_{NT}^*$ is differentiable at $\tilde{V}$ by Stokey and Lucas, 1989).

**Case 2** $c^*(\tilde{V}) = 0$. In this case, for all $V$, let

$$W(V) \equiv (\theta - u^{-1}(u(c(\tilde{V}))) + \max\{0, V - \tilde{V}\})) + \beta \int_{\Sigma} U_{NT}^*(V_r(\xi; \tilde{V}))dF(\xi).$$

The function $W$ such defined is concave, given that $u$ is increasing and concave.

For $V < \tilde{V}$, $W$ is differentiable with $W(V) = U_{NT}^*(\tilde{V}) \leq U_{NT}^*(V)$, given that $U_{NT}^*$ is decreasing by Step 3.

For $V > \tilde{V}$, $W$ is differentiable given that $u$ is differentiable, and $W(V) \leq U_{NT}^*(V)$ given that $c = u^{-1}(u(c^*(\tilde{V}))) + (V - \tilde{V})$ and $V_r(\xi; \tilde{V})$ for all $\xi$ satisfy (8)-(11).

For $V = \tilde{V}$, $W$ is left differentiable with $W'(V^-) = 0$, and right differentiable with $W'(V^+) = 0$ given $u'(0) = \infty$ by the Inada conditions. Thus $W$ is differentiable at $V$.

To summarize, $W$ is concave and differentiable with $W(V) \leq U_{NT}^*(V)$ for all $V$, and $W(\tilde{V}) = U_{NT}^*(\tilde{V})$. Apply again Stokey and Lucas (1989) and the desired result follows.

**Step 5** We show that $U_{NT}^*$ is strictly decreasing and strictly concave.
Let $\alpha(V)$, $\mu(V)$, and $\beta \gamma(\xi; V)$ be the Lagrangian multipliers for (8), (9), and (28) respectively. Then the Kuhn-Tucker conditions are as follows: for all $V$ and for all $\xi$,

$$-1 + \alpha(V)u'(c(V)) + \mu(V) = 0, \quad (29)$$

$$(U'_{NT}(V_r(\xi; V)) + \alpha(V))f(\xi) + \gamma(\xi; V) = 0, \quad (30)$$

$$\mu(V)c(V) = 0, \quad (31)$$

$$\gamma(\xi; V)(V_r(\xi; V) - \max\{\xi, \overline{V}\}) = 0, \quad (32)$$

$$\mu(V), \gamma(\xi; V) \geq 0. \quad (33)$$

Furthermore, the Envelope Theorem gives

$$U'_{NT}(V) = -\alpha(V). \quad (34)$$

Given that $U'_{NT}$ is concave by Step 3, (34) implies that $\alpha$ is increasing. To obtain the desired results for this step, we need only show that $\alpha$ is strictly increasing, which implies that $U'_{NT}$ is strictly concave by (34), which in turn implies that $U'_{NT}$ is strictly decreasing given that $U'_{NT}$ is decreasing by Step 3.

Suppose not. That is, suppose there exist $V^1, V^2 \in \Omega$ with $V^1 < V^2$ such that $\alpha(V) \equiv \tilde{\alpha}$, for some $\tilde{\alpha}$ and for all $V \in [V^1, V^2]$. Without loss of generality, suppose $V^1 = \inf\{V \in \Omega : \alpha(V) = \tilde{\alpha}\}$ and $V^2 = \sup\{V \in \Omega : \alpha(V) = \tilde{\alpha}\}$. We take the following three steps to derive a contradiction.

First, we show that for all $V \in [V^1, V^2]$, $c(V) \equiv \bar{c}$, for some $\bar{c}$.

Suppose $\tilde{\alpha} = 0$. Then for all $V \in [V^1, V^2]$, if $c(V) > 0$, then $\mu(V) = 0$ by (31), which implies that the LHS of (29) is $-1$, which contradicts with (29). We then conclude $c(V) = 0$.

Suppose $\tilde{\alpha} > 0$. Then for all $V \in [V^1, V^2]$, if $c(V) = 0$, then the LHS of (29) is $-1 + \tilde{\alpha}u'(0) + \mu(V) > 0$ by $u'(0) = \infty$ and $\mu(V) \geq 0$ by (33), which contradicts with (29). Thus $c(V) > 0$, which implies $\mu(V) = 0$ by (31), which in turn implies $c(V) = u^{-1}(1/\tilde{\alpha}) \equiv \bar{c}$ by (29) where, given that $u$ is strictly concave, the inverse function is well defined.

Second, we show that for all $V \in [V^1, V^2]$,

$$V_r(\xi; V) \begin{cases} \in [V^1, V^2], & \text{for } \xi \leq V^2 \\ = \xi, & \text{for } \xi > V^2 \end{cases}. \quad (35)$$

For all $\xi \leq V^2$, we show $\gamma(\xi; V) = 0$, which implies $\alpha(V_r(\xi; V)) = -U'_{NT}(V_r(\xi; V)) = \alpha(V) = \tilde{\alpha}$ by (30) and (34), which in turn implies $V_r(\xi; V) \in [V^1, V^2]$. Suppose $\gamma(\xi; V) > 0$. Then (32) implies $V_r(\xi; V) = \max\{\xi, \overline{V}\} \leq V^2$, which in turn implies that the LHS of (30) is strictly positive given that $\alpha$ is increasing, which contradicts with (30).
For all \( \xi > V^2 \), we show \( \gamma(\xi; V) > 0 \), which implies \( V_r(\xi; V) = \max\{\xi, V\} = \xi \) by (32). Suppose \( \gamma(\xi; V) = 0 \). Then (30) and (34) imply \( \alpha(V_r(\xi; V)) = \tilde{\alpha} \), which in turn implies \( V_r(\xi; V) \leq V^2 < \xi \), which contradicts with (4).

Third, we show \( V^1 = V^2 \), which implies that \( \alpha \) is strictly increasing.

Given (4) and (35), (8) implies
\[
\begin{align*}
  u(\tilde{c}) + \beta \left( F(V^1)V^1 + \int_{V^1}^{V_{\max}} \xi dF(\xi) \right) \leq V^1 \leq V^2 \leq u(\tilde{c}) + \beta \left( F(V^2)V^2 + \int_{V^2}^{V_{\max}} \xi dF(\xi) \right),
\end{align*}
\]
which in turn implies \( Y(V^1) \geq 0 \geq Y(V^2) \) where
\[
Y(V) \equiv V - \left[ u(\tilde{c}) + \beta \left( F(V)V + \int_{V}^{V_{\max}} \xi dF(\xi) \right) \right],
\]
which in turn implies \( V^1 \geq V^2 \) given \( Y'(V) = 1 - \beta F(V) > 0 \). Thus we conclude \( V^1 = V^2 \).

This completes Step 5 and hence the proof of Part (i) of the theorem. Part (ii) of the theorem follows from (35) and \( V^1 = V = V^2 \) by Step 5 from the above proof of Part (i). The proof of Theorem 1 is complete.

**C Proof of Corollary 1**

Given (ii) of Theorem 1, equation (8) implies that for all \( V \),
\[
\begin{align*}
u(c(V)) &= V - \beta \int_{\Sigma} \max\{\xi, V\} dF(\xi) < V - \beta V = (1 - \beta)V.
\end{align*}
\]
The lemma is proved.

**D Proof of Lemma 2**

**Step 1** We show that if \( V \in \Phi \), then \( V \geq u(0) + \beta E(\xi) \). Suppose \( V \in \Phi \). Then there exists a feasible contract \( \sigma = \{c(V'), I(\xi; V'), V_r(\xi; V'): \xi \in \Sigma \text{ and } V' \in \Phi \} \) satisfying (13)-(17) such that
\[
\begin{align*}
  V &= u(c(V)) + \beta \int_{\Sigma} [I(\xi; V)V_r(\xi; V) + (1 - I(\xi; V))\xi] dF(\xi) \\
  &\geq u(0) + \beta \int_{\Sigma} [I(\xi; V)\xi + (1 - I(\xi; V))\xi] dF(\xi) \\
  &= u(0) + \beta E(\xi),
\end{align*}
\]
where the first equality follows from (13), and the inequality follows from \( c(V) \geq 0 \) by (14) and \( V_r(\xi; V) \geq \xi \) for all \( \xi \in \Sigma \) by (16).
Step 2 We show that there is a feasible contract that attains all $V \in [u(0) + \beta E(\xi), V_{\max})$. This contract is constructed as follows. For all $\xi \in \Sigma$ and all $V \in [u(0) + \beta E(\xi), V_{\max})$, let

$$I(\xi; V) = 1, \text{ if } \xi < \bar{\xi}(V); \ 0, \text{ otherwise},$$

$$V_r(\xi; V) = \max\{\xi, V\},$$

where for all $V$, $\bar{\xi}(V) \in \Sigma$ is set, together with $c(V) \geq 0$, to be such that (13) holds. It is straightforward to show that such $c(V)$ and $\bar{\xi}(V) \in \Sigma$ exist and the lemma is proven.

E Proof of Theorem 2

By Lemma 2, constraints (22) and (23) can be combined to read as

$$V_r(\xi) \geq \max\{\xi, u(0) + \beta E(\xi)\}, \forall \xi \in \Sigma. \quad (36)$$

Proof of (i) & (ii) of the theorem Fix $V$. Let $\alpha$, $\mu$, $\beta \lambda(\xi)$, and $\beta \gamma(\xi)$ be the Lagrangian multipliers for (19)-(21) and (36) respectively. Then the Kuhn-Tucker conditions are as follows: for all $\xi$,

$$-1 + \alpha u'(c) + \mu = 0 \quad (37)$$

$$L(\xi)f(\xi) + \lambda(\xi)(1 - 2I(\xi)) = 0 \quad (38)$$

$$I(\xi)(U'(V_r(\xi)) + \alpha)f(\xi) + \gamma(\xi) = 0 \quad (39)$$

$$\mu c = 0 \quad (40)$$

$$\lambda(\xi)I(\xi)(1 - I(\xi)) = 0 \quad (41)$$

$$\gamma(\xi)(V_r(\xi) - \max\{\xi, u(0) + \beta E(\xi)\}) = 0 \quad (42)$$

$$\mu, \lambda(\xi), \gamma(\xi) \geq 0 \quad (43)$$

where

$$L(\xi) \equiv U(V_r(\xi)) - (U_0(U) - C_0) + \alpha(V_r(\xi) - \xi). \quad (44)$$

In addition, the Envelope Theorem gives

$$(\Gamma U)'(V) = -\alpha. \quad (45)$$

The remainder of the proof takes 4 steps, as organized in Lemmas 3-6, respectively.

Lemma 3 For all $\xi$, $I(\xi) = 1$ if $L(\xi) > 0$, and $I(\xi) = 0$ if $L(\xi) < 0$.  

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Proof. Fix $\xi$. Suppose $L(\xi) > 0$. Then (38) implies $\lambda(\xi)(1 - 2I(\xi)) < 0$, which in turn implies $\lambda(\xi) > 0$ and $1 - 2I(\xi) < 0$ given $\lambda(\xi) \geq 0$ by (43). Furthermore, $\lambda(\xi) > 0$ implies $I(\xi) = 0$ or $1$ by (41), and $1 - 2I(\xi) < 0$ implies $I(\xi) > 1/2$. Hence, we conclude $I(\xi) = 1$. The proof of the second part of Lemma 3 is similar. ■

In words, $L(\xi)$ is the net marginal benefit of retention given outside offer $\xi$.

**Lemma 4** Suppose that $U \in B(\Phi)$ is concave with $U'(u(0) + \beta E(\xi)) \geq 0$ and $U'(V) \leq 0$ for some $V$. Denote

$$V^*(U) \equiv \sup_{V \in \Phi} \left\{ \arg \max_{V \in \Phi} \{U(V)\} \right\} \in [u(0) + \beta E(\xi), V_{\text{max}}] \tag{12}$$

$$\overline{V}(U) \equiv u(0) + \beta \left( F(V^*(U))V^*(U) + \int_{V^*(U)}^{V_{\text{max}}} \xi dF(\xi) \right) \in (u(0) + \beta E(\xi), V_{\text{max}}).$$

Then the solution to the problem (37)-(43) has:

(i) For all $V$, there exist $\bar{\xi}_1 \in [V_{\text{min}}, V^*(U)]$, $\bar{\xi}_2 \in [V^*(U), V_{\text{max}}]$, and $\overline{V}_r \in [u(0) + \beta E(\xi), V_{\text{max}}]$ ($\bar{\xi}_1$, $\bar{\xi}_2$, and $\overline{V}_r$ depending on $V$) such that

$$I(\xi) = \begin{cases} 1 & \text{for } \xi \in (\bar{\xi}_1, \bar{\xi}_2) \\ 0 & \text{for } \xi < \bar{\xi}_1 \text{ and } \xi > \bar{\xi}_2 \end{cases};$$

$$V_r(\xi) = \max\{\xi, \overline{V}_r\}, \forall \xi. \tag{b}$$

(ii) $\Gamma U \in B(\Phi)$ is concave, and attains its maximum at $V = \overline{V}(U)$ with $(\Gamma U)'(u(0) + \beta E(\xi)) \geq 0$ and $(\Gamma U)'(V) \leq 0$ for some $V$.

Proof of (i) of Lemma 4 Fix $V$. We show that the desired result holds in the two cases $\alpha \geq 0$ and $\alpha < 0$ respectively.

**Case 1** $\alpha \leq 0$. The proof of (i) in this case takes four steps.

**Step 1** We show $c = 0$.

Equation (37) implies $\mu = 1 - \alpha u'(c) > 0$, which in turn implies $c = 0$ by (40).

**Step 2** We show that there exists $\overline{V}_r \in [u(0) + \beta E(\xi), V^*(U)]$ with $U'(\overline{V}_r) \leq -\alpha$ (where the equality holds if $\overline{V}_r > u(0) + \beta E(\xi)$) such that (b) holds.

Suppose $\int_\xi I(\xi)dF(\xi) = 0$. Then the choice of $V_r(\xi)$ is arbitrary for all $\xi$, and the existence of $\overline{V}_r \in [u(0) + \beta E(\xi), V^*(U)]$ with $U'(\overline{V}_r) \leq -\alpha$ is guaranteed by that $U$ is concave with $U'(u(0) + \beta E(\xi)) \geq 0$ and $U'(V^*(U)) = 0$.

Suppose $\int_\xi I(\xi)dF(\xi) > 0$. Then for all $\xi$ with $I(\xi) > 0$, if $\gamma(\xi) = 0$, then (39) implies $U'(V_r(\xi)) = -\alpha$; if $\gamma(\xi) > 0$, then (39) and (42) imply $V_r(\xi) = \max\{\xi, u(0) + \beta E(\xi)\}$ with $U'(V_r(\xi)) < -\alpha$. Hence, given that $U$ is concave, the result follows.

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\[\text{Note that } \arg \max_{V \in \Phi} \{U(V)\} \text{ is nonempty since } U \text{ is concave with } U'(V) \leq 0 \text{ for some } V.\]
Step 3 We show that there exist \( \xi_1 \in [V_{\min}, V^*(U)] \) and \( \xi_2 \in [V^*(U), V_{\max}] \) (\( \xi_2 \) is independent of \( V \)) such that (a) holds.

Given \( V_r(\xi) = \max\{\xi, \nabla_r\} \) for all \( \xi \) by Step 2, (44) implies

\[
L(\xi) = U(\max\{\xi, \nabla_r\}) - (U_0(U) - C_0) + \alpha(\max\{\xi, \nabla_r\} - \xi),
\]

which is increasing on \([V_{\min}, V^*(U)]\), and strictly decreasing on \([V^*(U), V_{\max}]\) with

\[
L(V^*(U)) = U(V^*(U)) - (U_0(U) - C_0) \geq 0
\]
given \( \nabla_r \leq V^*(U) \) by Step 2. The result then follows from Lemma 3.

Step 4 We show \( V \leq \nabla(U) \).

The result follows from Step 1-2 and (19).

Case 2 \( \alpha > 0 \). The proof of (i) in this case takes four steps.

Step 1 We show \( u'(c) = 1/\alpha \).

If \( c = 0 \), then \(-1 + \alpha u'(0) + \mu > 0 \) given \( u'(0) = \infty \) and \( \mu \geq 0 \) by (43), which contradicts with (37). We then proceed by assuming \( c > 0 \), which implies \( \mu = 0 \) by (40), which in turn implies \( u'(c) = 1/\alpha \) by (37).

Step 2 We show that there exists \( \nabla_r \in [V^*(U), V_{\max}] \) with \( U'(\nabla_r) \geq -\alpha \) (where the equality holds if \( \nabla_r < V_{\max} \)) such that (b) holds.

Suppose \( \int \Sigma I(\xi)dF(\xi) = 0 \). Then the choice of \( V_r(\xi) \) is arbitrary for all \( \xi \), and the existence of \( \nabla_r \in [V^*(U), V_{\max}] \) with \( U'(\nabla_r) \geq -\alpha \) is guaranteed by that \( U \) is concave with \( U'(V^*(U)) = 0 \).

Suppose \( \int \Sigma I(\xi)dF(\xi) > 0 \). Then for all \( \xi \) with \( I(\xi) > 0 \), if \( \gamma(\xi) = 0 \), then (39) implies \( U'(V_r(\xi)) = -\alpha \); if \( \gamma(\xi) > 0 \), then (39) and (42) imply \( V_r(\xi) = \max\{\xi, u(0) + \beta E(\xi)\} \) with \( U'(V_r(\xi)) < -\alpha \). Hence, given that \( U \) is concave, the result follows.

Step 3 We show that there exist \( \xi_1 = V_{\min} \) and \( \xi_2 \in [V^*(U), V_{\max}] \) such that (a) holds.

Given \( V_r(\xi) = \max\{\xi, \nabla_r\} \) for all \( \xi \) by Step 2, (44) implies

\[
L(\xi) = U(\max\{\xi, \nabla_r\}) - (U_0(U) - C_0) + \alpha(\max\{\xi, \nabla_r\} - \xi),
\]

which is strictly decreasing with

\[
L(V^*(U)) = U(\nabla_r) - (U_0(U) - C_0) + \alpha(\nabla_r - V^*(U))
\geq U(\nabla_r) - (U_0(U) - C_0) - U'(\nabla_r)(\nabla_r - V^*(U))
\geq U(V^*(U)) - (U_0(U) - C_0)
\geq 0,
\]

where the equality follows from \( \nabla_r \geq V^*(U) \) by Step 2, the first inequality follows from \( \nabla_r \geq V^*(U) \) and \( U'(\nabla_r) \geq -\alpha \) by Step 2, and the second inequality follows from that \( U \) is concave. The result then follows from Lemma 3.
**Step 4** We show $V > \nabla(U)$. This follows from Step 1-3 and (19).

**Proof of (ii) of Lemma 4** The proof takes two steps.

**Step 1** We show that $\Gamma U$ is increasing and concave on $[u(0) + \beta E(\xi), \nabla(U)]$.

Given what have been shown in Step 1-4 of Case 1, we have

$$(\xi_1 - V_{\min})L(\xi_1) = 0,$$

$$\left[\nabla_r - (u(0) + \beta E(\xi))]U'(\nabla_r) + \alpha\right] = 0,$$

$$u(0) + \beta \left(\int_{V_{\min}}^{\xi_1} \xi dF(\xi) + \int_{\xi_1}^{\xi_2} \max\{\xi, \nabla_r\} dF(\xi) + \int_{\xi_2}^{\nabla_{\max}} \xi dF(\xi)\right) = V^{13}.$$

Totally differentiating the equations above gives

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} \frac{d\xi_1}{dV} \\ \frac{d\nabla_r}{d\alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dV,$$

where

$$a_{11} = L(\xi_1) - \alpha(\xi_1 - V_{\min}) \geq 0,$$

$$a_{12} = (\xi_1 - V_{\min})(U'(\nabla_r) + \alpha) \leq 0,$$

$$a_{13} = (\xi_1 - V_{\min})(\nabla_r - \xi_1) \geq 0,$$

$$a_{22} = (U'(\nabla_r) + \alpha) + U''(\nabla_r)[\nabla_r - (u(0) + \beta E(\xi))] \leq 0,$$

$$a_{23} = \nabla_r - (u(0) + \beta E(\xi)) \geq 0,$$

$$a_{31} = -\beta f(\xi_1)(\nabla_r - \xi_1) \leq 0,$$

$$a_{32} = \beta (F(\nabla_r) - F(\xi_1)) \geq 0.$$

Furthermore, by Cramer’s rule,

$$\frac{d\xi_1}{dV} = \frac{\|A_1\|}{\|A\|} \leq 0, \quad \frac{d\nabla_r}{d\alpha} = \frac{\|A_2\|}{\|A\|} \geq 0, \text{ and } \frac{d\alpha}{dV} = \frac{\|A_3\|}{\|A\|} \geq 0,$$

where

$$\|A\| = -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \leq 0,$$

$$\|A_1\| = -a_{13}a_{22} \geq 0,$$

$$\|A_2\| = -a_{11}a_{23} \leq 0,$$

$$\|A_3\| = a_{11}a_{22} \leq 0.$$

---

For $V \in [u(0) + \beta E(\xi), \nabla(U)], \xi_2$ is independent of $V$ by Step 3 of Case 1. Therefore, it can be treated as a constant. In addition, it is straightforward to show $\bar{\xi}_1 \leq \nabla_r \leq \bar{\xi}_2$. 

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Thus, $\Gamma U$ is concave given $(\Gamma U)''(V) = -d\alpha/dV \leq 0$ by (45).

**Step 2** We show that $\Gamma U$ is strictly decreasing and concave on $(V(U), V_{max})$.

Given what have been shown in Step 1-4 of Case 2, we have

$$L(\xi_2)(\xi_2 - V_{max}) = 0,$$

$$(U'(V_r) + \alpha)(V_r - V_{max}) = 0,$$

$$u\left(u^{-1}\left(\frac{1}{\alpha}\right)\right) + \beta \left(\int_{V_{min}}^{\xi_2} \max\{\xi, V_r\} dF(\xi) + \int_{\xi_2}^{V_{max}} \xi dF(\xi)\right) = V.$$

Totally differentiating the equations above gives

$$\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  0 & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  \frac{d\xi_2}{dV_r} \\
  \frac{dV_r}{d\alpha} \\
  \frac{d\alpha}{dV}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  1
\end{bmatrix}$$

where

- $a_{11} = \min\{-\alpha, U'(\xi_2)\}(\xi_2 - V_{max}) + L(\xi_2) \geq 0$,
- $a_{12} = (U'(V_r) + \alpha)(\xi_2 - V_{max}) \leq 0$,
- $a_{13} = (\max\{\xi_2, V_r\} - \xi_2)(\xi_2 - V_{max}) \leq 0$,
- $a_{22} = U''(V_r)(V_r - V_{max}) + (U'(V_r) + \alpha) \geq 0$,
- $a_{23} = V_r - V_{max} \leq 0$,
- $a_{31} = \beta f(\xi_2)(\max\{\xi_2, V_r\} - \xi_2) \geq 0$,
- $a_{32} = \beta F(\min\{\xi_2, V_r\}) \geq 0$,
- $a_{33} = -u'(c)/(\alpha^2 u''(c)) \geq 0$.

Furthermore, by Cramer’s rule,

$$\frac{d\xi_2}{dV} = \frac{\|A_1\|}{\|A\|} \geq 0, \quad \frac{dV_r}{dV} = \frac{\|A_2\|}{\|A\|} \geq 0, \quad \text{and} \quad \frac{d\alpha}{dV} = \frac{\|A_3\|}{\|A\|} \geq 0,$$

where

- $\|A\| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{13}a_{22}a_{31} \geq 0$,
- $\|A_1\| = -a_{13}a_{22} \geq 0$,
- $\|A_2\| = -a_{11}a_{23} \geq 0$,
- $\|A_3\| = a_{11}a_{22} \geq 0$.

Thus, $\Gamma U$ is concave given $(\Gamma U)''(V) = -d\alpha/dV$ by (45).
To summarize, $\Gamma U$ is concave, and for all $V \in \Phi$,
\[
\frac{d\xi_1}{dV} \leq 0, \quad \frac{d\xi_2}{dV} \geq 0, \text{ and } \frac{dV_r}{dV} \geq 0.
\tag{46}
\]

\[\blacksquare\]

**Lemma 5** Let $U_0 = \overline{U} \in B(\Phi)$ and $U_{n+1} = \Gamma U_n$ for $n = 0, 1, \ldots$. Then the sequence $\{U_n\}_{n=0}^{\infty} \subseteq B(\Phi)$ converges pointwisely and monotonically to the principal’s value function $U^*$ as $n$ goes to infinity. In addition, $V^*(\Gamma^\infty U) = \nabla(\Gamma^\infty U) = \nabla$. 

**Proof.** Given that the mapping $\Gamma$ is monotonic and preserves concavity by (ii) of Lemma 4, it is straightforward to show that $\Gamma^n U$ is concave with $U(V) \geq \Gamma^n U(V) \geq \Gamma^{n+1} U(V) \geq U^*(V)$ for all $V$ and $n = 1, 2, \ldots$. The result then follows. In addition, $V^*(\Gamma^\infty U) = \nabla(\Gamma^\infty U) = \nabla$. 

**Lemma 6** For all $V$, $V_r = V$. 

**Proof.** Fix $V$. The proof takes two steps.

**Step 1** We show $U''(V) = U''(V_r) = -\alpha$.

The result follows from (45) and Step 2 of Case 1-2 of the proof of (i) of Lemma 4.

**Step 2** We show $V_r = V$.

By Step 1, $U''(V) = U''(V_r(V)) = -\alpha$ for $t = 1, \ldots$ where $V_r(V)$ denotes the agent’s continuation expected utility starting from period $t + 1$ conditional on retention and $\xi \leq V_r(V)$ in period $\tau = 1, \ldots, t$. It implies $c(V) = c(V_r(V)) \equiv c$ by Step 1 of the proof of (i) of Lemma 4, $\bar{\xi}_1(V) = \bar{\xi}_1(\nabla^n_r(V)) \equiv \bar{\xi}_1$, and $\bar{\xi}_2(V) = \bar{\xi}_2(\nabla^n_r(V)) \equiv \bar{\xi}_2$ by (44) and Lemma 3. Hence, for $t = 1, \ldots, (19)$ implies

$$u(c) + \beta \left( \int_{V_{\min}}^{\bar{\xi}_1} \xi dF(\xi) + \int_{\bar{\xi}_1}^{\bar{\xi}_2} \max\{\xi, V_r(V)\} dF(\xi) + \int_{V_{\max}}^{\bar{\xi}_2} \xi dF(\xi) \right) = \nabla_r(V)^{t-1},$$

where $V \equiv V_r^0(V)$. Suppose $V \neq V_r(V)$. Given $\beta \in (0, 1)$, $\{V_r(V)^t\}_{t=1}^{\infty}$ is either an increasing sequence converging to $\infty > V_{\max}$, or a decreasing sequence converging to $-\infty < V$. Therefore, we conclude $V = V_r(V)$. This completes the proof of Lemma 6 and hence the proof of parts (i) and (ii) of the theorem. 

We now proceed to prove the last part of the theorem. Given (ii) of the theorem which we have just proved, the promise-keeping constraint (19) can be rewritten as

$$u(c) + \beta \left( \int_{V_{\min}}^{\bar{\xi}_1} \xi dF(\xi) + \int_{\bar{\xi}_1}^{\bar{\xi}_2} \max\{\xi, V_r(V)\} dF(\xi) + \int_{V_{\max}}^{\bar{\xi}_2} \xi dF(\xi) \right) = V; \quad (47)$$
the net marginal benefit of retention (44) can be rewritten as
\[ L(\xi;V) = U^*(\max\{\xi,V\}) - (U_0(U^*) - C_0) + \alpha(\max\{\xi,V\} - \xi); \] (48)
and the value function (18) can be rewritten as
\[ U^*(V) = (\theta - c) + \beta \left\{ \int_{\xi_1}^{\xi_2} U^*(\max\{\xi,V\})dF(\xi) + [F(\xi_1) + (1 - F(\xi_2))](U_0(U^*) - C_0) \right\}. \] (49)

**Proof of (iii) of the theorem** Let \( V < \bar{V} \). Given \( \alpha = -U''(V) \) by (45), (47) implies
\[ L(\xi) = U^*(\max\{\xi,V\}) - (U_0(U^*) - C_0) - U''(V)(\max\{\xi,V\} - \xi), \]
as illustrated in the Figure 4.

For \( \xi \geq V \), \( L(\xi) = U^*(\xi) - (U_0(U^*) - C_0) \). This part of the function \( L \) is depicted by the curved line in the figure. For \( \xi < V \), \( L(\xi) = U^*(V) - (U_0(U^*) - C_0) - U''(V)(V - \xi) \). This part of \( L \) is depicted by the straight line (with a constant slope of \( U''(V) \)) in the figure. Apparently, \( L \) is continuous at \( V \).

![Figure 4: The Retention/Termination Region for \( V < \bar{V} \)](image)

Given \( c = 0 \) by Step 1 of Case 1 of the proof of (i) of Lemma 4 and \( \xi_2 \geq \bar{V} > V \) by (ii) of Theorem 2, (47) implies
\[ u(c) + \beta \left( \int_{V_{\min}}^{\xi_1} \xi dF(\xi) + \int_{\xi_1}^{V_{\max}} \max\{\xi,V\}dF(\xi) \right) = V. \]
Suppose $\xi_1 = V_{\text{min}}$. Then $V = V_{\text{min}}$ by (6), which contradicts with $V < V_{\text{min}}$. Hence, we conclude $\xi_1 > V_{\text{min}}$. Moreover, given (46) and (ii) of Theorem 2, and as is obvious from the figure, $\xi_1$ as a function of $V$ is decreasing, as long as $V < V_{\text{min}}$ (and strictly decreasing if $V$ is sufficiently small so that $L(\xi)$ cuts the horizontal line at a level of $\xi$ great than $V_{\text{min}}$).

Last, suppose $V \geq V_{\text{min}}$. Then the slope of the function $L$ over the interval $[V_{\text{min}}, V]$ would be negative and it should be obvious that $\xi_1$ should be constant at $V_{\text{min}}$.

Proof of (iv) of the theorem Fix $V$. The proof takes three steps.

Step 1 We show that with the optimal contract, an outside offer $\xi \geq V_{\text{min}}$ is matched if and only if $U^*(\xi) \geq U_0(U^*) - C_0$.

The only if part is obvious. The if part: For all $\xi \geq V_{\text{min}}$, (48) implies $L(\xi; V) = U^*(\xi) - (U_0(U^*) - C_0)$. The desired result follows by applying Lemma 3.

Step 2 We show that for any $\xi \geq V_{\text{min}}$, the inequality $U^*(\xi) \geq U_0(U^*) - C_0$ holds if and only if $\xi \leq V''$.

Consider $V = u(0) + \beta E(\xi)$. This value of the agent’s expected utility can be achieved with a feasible contract that gives the agent first period consumption $c(V) = 0$, first period probability of retention $I(\xi, V) = 0$, and $V_r(\xi) = \max \{\xi, V\}$: $\forall \xi$. But this implies

$$U^*(u(0) + \beta E(\xi)) \geq \theta + \beta (U_0(U^*) - C_0) \geq U_0(U^*) - C_0.$$ 

The desired result then follows from the fact that $U^*$ is concave by (i) of the theorem, which was already proven, and the definition of $V''$.

Step 3 We show that $\xi_2$ as a function of $V$ is increasing, but constant at $V''$ for $V \leq V''$. This follows directly from (46) and Step 2.

The proof of Theorem 2 is now complete.

F Proof of Corollary 2

Observe that for all $V \leq V_{\text{min}}$, $u(c) = u(0) \leq (1 - \beta)V$, by Step 1 of Case 1 of the proof of (i) of Lemma 4. Observe next that for all $V > V_{\text{min}}$, given $\bar{\xi}_1 = V_{\text{min}}$ by (iii) of Theorem 2, (47) implies

$$u(c) + \beta \left( \int_{V_{\text{min}}}^{\xi_2} \max \{\xi, V\} dF(\xi) + \int_{\xi_2}^{V_{\text{max}}} \xi dF(\xi) \right) = V.$$ 

(50)

Suppose $\xi_2 \geq V$. Then (50) implies

$$u(c) = V - \beta \left( \int_{V_{\text{min}}}^{\xi_2} \max \{\xi, V\} dF(\xi) + \int_{\xi_2}^{\xi_2} \xi dF(\xi) \right)$$

$$= V - \beta \int_{\Sigma} \max \{\xi, V\} dF(\xi)$$

$$< (1 - \beta)V.$$
where the second equality follows from $\xi = \max\{\xi, V\}$ for all $\xi \geq \bar{\xi}_2$ given $\bar{\xi}_2 \geq V$.

Suppose $\bar{\xi}_2 < V$. Then (50) implies
\[
u(c) + \beta \left( F(\bar{\xi}_2) V + \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi) \right) = V.
\]
Furthermore, (48) implies
\[L(\bar{\xi}_2; V) = U^*(V) - (U_0(U^*) - C_0) + (1/u'(c))(V - \bar{\xi}_2) = 0,
\]
given $\bar{\xi}_2 < V$ and $\alpha = 1/u'(c)$ by Step 1 of Case 2 of the proof of (i) of Lemma 4. And (49) implies
\[
U^*(V) = \frac{(\theta - c) + \beta(1 - F(\bar{\xi}_2))(U_0(U^*) - C_0)}{1 - \beta F(\bar{\xi}_2)}
\]
given $\bar{\xi}_1 = V_{\min}$ by (iii) of Theorem 2 and $\bar{\xi}_2 < V$. With these, (51) implies
\[
0 = \nu(c) - (1 - \beta F(\bar{\xi}_2)) V + \beta \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi)
\]
\[
= \nu(c) - (1 - \beta F(\bar{\xi}_2))\{\bar{\xi}_2 - u'(c)[U^*(V) - (U_0(U^*) - C_0)]\} + \beta \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi)
\]
\[
= \nu(c) + u'(c)[(\theta - c) - (1 - \beta)(U_0(U^*) - C_0)] - (1 - \beta F(\bar{\xi}_2))\bar{\xi}_2 + \beta \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi)
\]
\[
\geq \nu(c) - u'(c)c - (1 - \beta F(\bar{\xi}_2))\bar{\xi}_2 + \beta \int_{\bar{\xi}_2}^{V_{\max}} \xi dF(\xi)
\]
\[
> \nu(c) - u'(c)c - (1 - \beta F(V)) V + \beta \int_{V}^{V_{\max}} \xi dF(\xi),
\]
where the second equality follows from (52), the third equality follows from (53), the first inequality follows from $U_0(U^*) - C_0 \leq \theta/(1 - \beta)$, and the second inequality follows from that the last two terms together on the fourth line is a strictly decreasing function of $\bar{\xi}_2$ and $\bar{\xi}_2 < V$.

Furthermore, given that $\nu(c) - u'(c)c$ is a strictly increasing function of $c$, $\nu(c) < (1 - \beta) V$ if $K(V) \geq 0$ where
\[
K(V) \equiv \nu(u^{-1}((1 - \beta) V)) - u'(u^{-1}((1 - \beta) V)) u^{-1}((1 - \beta) V) - (1 - \beta F(V)) V + \beta \int_{V}^{V_{\max}} \xi dF(\xi).
\]

In fact, we have $K(V) \geq 0$ for all $V > \bar{V}$. This holds because
\[
K'(V) = (1 - \beta) \left( -\frac{u''(u^{-1}((1 - \beta) V))u^{-1}((1 - \beta) V)}{u'(u^{-1}((1 - \beta) V))} \right) - (1 - \beta F(V))
\]
\[
\geq \beta F(V)
\]
\[
\geq 0,
\]

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where the first inequality follows from that the relative risk aversion coefficient is greater than $1/(1 - \beta)$ for all $c \geq u^{-1}((1 - \beta)V)$ and $V > \overline{V}$, and

\[
K(\overline{V}) = u(u^{-1}((1 - \beta)\overline{V})) - u'(u^{-1}((1 - \beta)\overline{V}))u^{-1}((1 - \beta)\overline{V}) - (1 - \beta F(\overline{V}))\overline{V} + \beta \int_{\overline{V}}^{V_{\text{max}}} \xi dF(\xi)
\]

\[
= u(u^{-1}((1 - \beta)\overline{V})) - u'(u^{-1}((1 - \beta)\overline{V}))u^{-1}((1 - \beta)\overline{V}) - u(0)
\]

\[
\geq 0,
\]

where the second equality follows from (6), and the inequality follows from that $u$ is concave. This completes the proof of the corollary.

References


