

# Supply Management in Multi-Product Firms with Fixed Proportions Technology

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## **Abstract**

This paper provides insights on the optimal procurement decisions of a processor for sourcing a primary commodity input, where this input gives rise to two outputs in fixed proportions. The processor can source the input using two heterogeneous option contracts, signed in advance of the spot market, and from the spot market on the spot day; and faces a newsvendor problem in each of the output markets. Our analysis provides managerial insights on the impact of input spot price and output demand uncertainties. We show that the processor benefits from a higher demand correlation, a lower demand variability, and a higher input spot price variability. Our comparative statics results provide some rules of thumb for strategic supply management. We show that some of these insights critically depend on the contract structure and the spot procurement access of the processor. For example, with a higher demand correlation, the processor that has spot procurement access should decrease its contract volume with a forward contract, but may increase it with an option contract. In the absence of spot procurement, the optimal volume of forward contract is insensitive to demand correlation. We demonstrate that the optimal contracting decision from the two option contracts follows a greedy policy under deterministic demands: The contracts are ranked in decreasing order of an index, which we characterize in closed form, and the second contract is used only after the capacity of the first contract is exhausted. With stochastic demands, the greedy policy is optimal only under special conditions.

**Key Words:** Contracting, Commodity Risk Management, Multi-product Newsvendor, Option Contracts, Spot Market.

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# 1 Introduction and Literature Review

The objective of this paper is to develop a theoretical basis for understanding the tradeoffs facing a commodity processor in the choice of alternative arrangements for sourcing a primary input, where this input gives rise to multiple outputs in fixed proportions, and when the processor acts as a wholesaler into these output markets. The risk management and contracting problem considered here is relevant for several commodity markets, including many agricultural products. For example, in cocoa markets, cocoa beans are processed to produce cocoa butter and cocoa powder. In sugar markets, sugarcane is processed to produce white table sugar (sucrose) and animal feed. In wheat markets, wheat seeds are processed to produce wheat bran and course powder flour. In all of these markets, a primary commodity input (cocoa bean, sugarcane, wheat seed) is processed into multiple outputs in fixed proportions.

There are unique characteristics of these commodity markets both on the upstream market side as well as on the downstream market side that present challenges for the processor's supply management. On the upstream market between the processor and the suppliers, there are two markets of interest: the contract market and the spot market.

Contract markets feature longer-term arrangements between the processor and the upstream suppliers. These contracts may take different forms in terms of pricing and delivery requirement. In this paper, we focus on the option contract, which is one of the most common contract forms used in commodity markets. An option contract specifies the volume of input reserved in advance of the spot market (reservation stage). The processor decides the actual delivery volume within this reserved capacity on the spot day (execution stage). This contract form also encompasses forward contracts where all the contracted volume is delivered to the processor.

Spot markets are regional markets primarily used as a topping up of contracts on the spot day. In line with agricultural commodity markets, our target application, we will assume throughout that spot markets are competitive, i.e. the price is not sensitive to the actions of any of the agents (Buyers or Sellers) who participate in this market.

On the downstream market between the processor and the wholesalers, we consider a multi-product, fixed proportions technology model where each unit of upstream product (input) yields a certain number of saleable downstream products (outputs) in fixed propor-

tions. The appropriate product market model depends on whether the outputs are traded commodities or not, and the substitution possibilities available in these output markets. In this paper, we assume that the commodity processor faces a newsvendor problem in each of the output markets, i.e. the output prices are fixed and the demands are stochastic; in a single period and there are no substitution possibilities. The model considered here is relevant for several agricultural markets (including cocoa, sugar and wheat markets), several chemical product markets, as well as for regulated product markets where the price is controlled by an entity. This modeling choice is motivated by our conversations with the procurement managers of one of the major multinational corporations<sup>1</sup> that procures a wide range of final products from several commodity processors. For the non-traded agricultural products<sup>2</sup> (such as cocoa powder and cocoa butter) as well as for the majority of chemical products (such as enzymes, colorants, preservatives etc.), a fixed-price contract with minimum and maximum delivery limits is the most common procurement contract used by this firm. The contract price is fixed for 3 to 6 months, which is the time frame considered in our single-period model.<sup>3</sup> The actual procurement volume is determined on the delivery date, but minimum and maximum delivery volumes are specified when these contracts are signed. Therefore, the commodity processors face stochastic demand for their final products paralleling the newsvendor model considered in this paper.

Against the background of these characteristics, we model a processor who procures a primary input, produces and sells two outputs in fixed proportions of this input in a single period. The processor can source the input using two heterogeneous option contracts in advance of the spot market. These contracts are characterized by their reservation price, execution price and the maximum capacity available. The processor chooses the contract volume to reserve from each contract to maximize its expected profit in the presence of input spot price and output demand uncertainties. After these uncertainties are realized,

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<sup>1</sup>This multinational corporation owns many of the world's consumer products brands in foods, beverages, cleaning agents and personal care products.

<sup>2</sup>For the traded agricultural products such as palm oil, spot price benchmarked contracts are used instead of fixed-price contracts. For these product markets, our model is only applicable for small processors that do not have an access to spot markets and that use fixed-price contracts.

<sup>3</sup>The fixed-price contract has several implementation advantages for this firm both on the upstream part (for example, it is easier to set up invoice systems with processors) and on the downstream part (for example, it is easier to set an internal transfer price and market prices for its own products as its customers do not expect to see varying prices).

the processor decides the volume of input to exercise from each contract, the volume of spot procurement, and the production quantity of each output within its available yield through fixed proportions technology.

We characterize the optimal contracting decision of the processor. We investigate how the input spot price uncertainty and the output demand uncertainty impact the optimal expected profit and the optimal contract volume of the processor. Our analysis focuses on determining the impact of the contract structure (for example, forward versus option contracts) and the spot procurement access of the processor on these results. This analysis is of practical importance for the commodity processors as empirical observations document different sourcing portfolios (different contract forms with or without spot procurement). The difference between these sourcing portfolios concerning the impact of spot price and demand uncertainties is not well-understood. Our results contribute to several streams of research, as detailed below.

Our primary contribution is to the literature on contracting in the presence of spot markets. This stream of papers provide conditions under which long-term contracting is a part of the optimal sourcing portfolio. For example, Mendelson and Tunca (2007) provide a rationale for the existence of forward contracts, based on strategic spot trading. Secomandi and Kekre (2011) demonstrate that forward contracts are beneficial when transaction costs of spot procurement are higher than that of contract procurement. We refer the readers to Kleindorfer and Wu (2003) for a review of papers in this area.

A recent stream of papers in this literature (Kazaz 2004, Goel and Gutierrez 2006, Devalkar et al. 2007, Kouvelis et al. 2010, Secomandi 2010) analyzes the joint sourcing, processing and production/pricing decisions of commodity processors. All these papers focus on a single-product setting where fixed proportions technology is irrelevant, and use a single contract for procurement. Closest to our work, Boyabatlı et al. (2011) analyze the optimal procurement, processing and production decisions of a meatpacker in beef supply chains, where the meatpacker processes fed cattle to produce two beef-products in fixed proportions. The meatpacker's procurement portfolio consists of spot procurement and a single quantity commitment contract whose price is benchmarked on the prevailing spot price. In this paper, motivated by different agricultural commodity markets, we use a different product market modeling approach, and include option contracts as well as multiple contracts with heterogeneous parameters. We provide new comparative statics

results and delineate the impact of the contract structure and the spot procurement access of the processor on these results. We show that there exists a subtle interplay between these two features: With spot procurement access, when the processor uses a forward contract, a higher demand correlation does not affect the optimal contract volume, whereas, with option contracts, a higher correlation increases the optimal contract volume. Without spot procurement access, the latter result holds only if the exercise price of the option contract is sufficiently high. Otherwise, a higher correlation decreases the optimal contract volume. These results provide some rules of thumb for the strategic supply management.

In the literature on contracting in the presence of spot markets, only a few papers (Kleindorfer and Wu 2003, Martínez-de-Albeniz and Simchi-Levi 2006) analyze the optimal sourcing decision with multiple contracts. These papers focus on a single-product setting where fixed proportions technology is irrelevant. With deterministic demand, Kleindorfer and Wu (2003) demonstrate that the optimal contracting decision follows a greedy policy: The contracts are ranked in decreasing order of an index and the lower-ranked contract is used only after the capacity of the higher-ranked contract is exhausted. We extend this result to multiple-output setting with fixed proportions technology, and characterize a closed-form contract index. We show that, with stochastic demands, the optimal contracting decision is more complex, and follows a greedy policy only under special conditions. Under the greedy procurement policy, the order of the contracts is determined by two indices, which we also characterize in closed form.

Two streams of literature are related to the multiple-output character of this paper. The first stream is the literature on newsvendor network models. As defined in Van Mieghem and Rudi (2002), newsvendor networks encompass the structural properties of the single-product newsvendor problem and extend this to the multi-product setting. We refer the reader to Dong et al. (2010) for a review of papers using a newsvendor network formulation and for further applications in the context of multiple markets and transshipment networks. The fixed proportions technology requires a generalization of the newsvendor network model to include proportional input-output relationships. A special case of our model (when the processor uses a forward contract and does not have spot procurement access) simplifies to a single-input, multiple-output newsvendor with fixed proportions technology. In this case, we show that the critical fractile solution of the single-input, single-output newsvendor problem extends to our setting with effective underage and effective overage costs that arise from the

fixed proportions technology. By analyzing option contracts and allowing spot procurement, we provide additional generalizations to the newsvendor network model, and provide new insights. Van Mieghem and Rudi (2002) demonstrate that in a single-input, multiple-output newsvendor problem without the fixed proportions technology, the expected profit *decreases* in demand correlation due to a lower value of capacity pooling. With the fixed proportions technology, we show that the processor's expected profit *increases* in demand correlation due to a higher value of input cost pooling.

A second stream of papers analyzes coproduction systems where multiple outputs are produced simultaneously in a single production run. The standard coproduction problem foresees different grades or quality levels of output, where yields for these different grades are typically random. The problem of contracting for inputs (e.g., wafer starts in semiconductor manufacture) when facing demand schedules for each of the grades has some similarities to the processing problem considered in this paper. However, the primary focus in the coproduction literature is on the production quantity and the allocation of the realized production output to the product demands. Several scholars investigate the structural properties of the optimal solution (e.g., Gerchak et al. 1996) and further propose heuristic solutions under deterministic demand (Bitran and Dasu 1992, and Bitran and Gilbert 1994) and stochastic demand (Hsu and Bassok 1999). More recently, Tomlin and Wang (2008) focus on a demand model of utility-maximizing customers in a single-period, two-product, two-customer class coproduction system. They solve for the optimal production, pricing, allocation decisions and analyze the value of different operational flexibilities (price postponement, allocation postponement and downward substitution). In contrast, the focus of our paper is the integration of sourcing, processing and production decisions with non-random yields facing input spot price and output demand uncertainties.

The remainder of this paper is organized as follows: In §2, we describe the model and discuss the basis for our assumptions. §3 provides the optimal solution for the firm's sourcing, processing and production decisions at the spot market stage. In §4, we characterize the optimal contracting decision with a single contract, and investigate the impact of input spot price and output demand uncertainties on the optimal expected profit and the optimal contract volume of the firm. §5 provides the optimal solution for the contracting decision with two contracts. We conclude in §6 with a discussion of the limitations of our analysis and future research directions.

## 2 Model Description

**Notation and Preliminaries.** A realization of the random variable  $\tilde{y}$  is denoted by  $y$ . Bold face letters represent vectors of the required size. Vectors are column vectors and  $'$  denotes the transpose operator.  $\frac{\mathbf{f}}{\mathbf{g}}$  and  $\mathbf{fg}$  denote the componentwise quotient and product of the vectors  $\mathbf{f}$  and  $\mathbf{g}$  with identical dimensions, respectively. We have  $(x)^+ = \max(x, 0)$ . The monotonic relations (increasing, decreasing) are used in the weak sense unless otherwise stated. For notational convenience, we use the superscript  $i$  to denote the index of the contract and the subscript  $j$  to denote the index of the final product.

We consider a firm that procures and processes a single primary input to produce two final products. We model the firm's procurement, processing and production decisions in a two-period framework. Before discussing the timeline in detail, we introduce our assumptions about the firms' decisions.

For the procurement decision, in line with the above discussion, we consider two sources: option contracts and the spot market. A typical contract specifies the volume of input reserved by the firm in advance of the spot market. On the spot day, the firm decides the volume of input to be delivered within the reserved capacity. We assume that there are two different contracts available for the firm. The contract  $i = 1, 2$  is characterized by the maximum volume of input  $\bar{Q}^i$  that can be reserved, the unit reservation price  $\beta^i$  that is paid in advance of the delivery, and the unit exercise price  $b^i$  that is paid upon delivery on the spot day. Let  $Q^i$  denote the volume of input reserved from contract  $i$  in advance of the spot market, and  $z^i$  denote the volume of input processed from contract  $i$  within the reserved capacity  $Q^i$  on the spot day. Let  $\bar{\mathbf{Q}}' \doteq (\bar{Q}^1, \bar{Q}^2)$ ,  $\mathbf{Q}' \doteq (Q^1, Q^2)$ ,  $\mathbf{z}^{\mathbf{C}'} \doteq (z^1, z^2)$  denote the vector of the contract capacity, the reserved input and the exercised input from the contract, respectively.

The firm can also buy input from the spot market on the day at the prevailing spot price  $P^S$ . Any transaction cost of the spot procurement is embedded in this spot price. We assume that  $\tilde{P}^S$  has a continuous distribution with positive support and finite expectation  $\mu_S$  and standard deviation  $\sigma_S$ . To analyze the two special cases of spot procurement in one model, we assume a capacity limit  $K$  on the spot procurement volume. Throughout the paper, we either assume  $K = 0$ , i.e. the firm does not have spot procurement access; or  $K \rightarrow \infty$ , i.e. the firm has unlimited spot procurement access. We make comparisons between these two special cases to delineate the impact of spot procurement access on our

main results. Let  $z^S$  denote the volume of input processed via spot procurement.<sup>4</sup>

For the processing and production decisions, we assume a unit processing cost of  $\omega$  whatever the source of the input, i.e. the total processing cost is given by  $\omega(\mathbf{1}'\mathbf{z}^C + z^S)$  for a given processed input  $(\mathbf{z}^C, z^S)$ . There are  $j = 1, 2$  possible outputs. Let  $\mathbf{a}' = (a_1, a_2)$  denote the fixed proportion vector such that for a given processed input  $(\mathbf{z}^C, z^S)$ , the available yield for product  $j$  is  $a_j(\mathbf{1}'\mathbf{z}^C + z^S)$ . The demands for the final products are stochastic, correlated and are denoted by  $\tilde{\mathbf{D}}' = (\tilde{D}_1, \tilde{D}_2)$ .  $\tilde{\mathbf{D}}$  is a bivariate random variable with a continuous distribution that has bounded expectation  $(\mu_1, \mu_2)$  with covariance matrix  $\Sigma$ , where  $\Sigma_{jj} = \sigma_j^2$  for  $j = 1, 2$  and  $\Sigma_{12} = \rho\sigma_1\sigma_2$  and  $\rho$  denotes the correlation coefficient between  $\tilde{D}_1$  and  $\tilde{D}_2$ . The available yield  $a_j(\mathbf{1}'\mathbf{z}^C + z^S)$  can either be converted into the final product  $j$  by incurring a production cost  $c_j > 0$ , or can be salvaged as a semi-processed input (without incurring the production cost) from a unit price of  $\bar{s}_j \geq 0$ . Let  $\mathbf{x}' = (x_1, x_2)$  denote the production vector. After the production, the firm can sell product  $j$  from a unit price  $p_j > 0$  if there is unsatisfied demand  $D_j$ , otherwise the unit is salvaged from  $\underline{s}_j \geq 0$ . The firm incurs a penalty cost  $r_j \geq 0$  for each unit of the unsatisfied demand for product  $j$ . Let  $\mathbf{p}, \underline{\mathbf{s}}, \bar{\mathbf{s}}, \mathbf{r}, \mathbf{c}$  denote the price, salvage value for the product, salvage value for the semi-processed input, penalty and production cost vector, respectively. We assume that  $p_j \geq \bar{s}_j$ , i.e. if there is no production cost, the firm prefers to sell the final product over salvaging the semi-processed input. For brevity, we assume  $a_1 + a_2 = 1$ , i.e. there is no yield loss in processing. The yield loss in processing can easily be incorporated in the model<sup>5</sup>; and all of our results continue to hold with minor modifications in the analysis.

We note here that the single commodity input, two-output model with the fixed proportions technology described above is suitable for several agricultural and chemical product markets. Our main motivation comes from the agricultural markets, thus we will base our discussion on these markets to give specific examples for the practical relevance of our model.

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<sup>4</sup>It is theoretically possible that a profit-maximizing processor could sell the contracted units to the spot market if the spot price is sufficiently high such that spot resale is more profitable than processing. Therefore, a more general model would allow the processor to use spot markets both for procurement and resale. However, this theoretical possibility is not of great interest in practice, as agricultural commodity processors hardly use spot markets for resale (see for example, Boyabath et al. (2011) in the context of the beef markets). Thus, to avoid uninteresting complications, we assume that there is no spot resale of the primary input. This enables us to focus on the procurement impact of spot markets.

<sup>5</sup>In particular, we can define a dummy product with index  $j = 0$  with fixed proportion  $a_0 = 1 - (a_1 + a_2)$  with its demand, and cost and revenue parameters normalized to 0, i.e.  $D_0 = p_0 = c_0 = r_0 = \bar{s}_0 = \underline{s}_0 = 0$ .

In cocoa markets, cocoa beans are processed (by cleaning, roasting and grinding) to produce cocoa liquor. The cocoa processor can either sell the cocoa liquor (as a semi-processed input) to chocolate manufacturers or further process the liquor (by pressing and milling) to produce cocoa butter and cocoa powder. In sugar markets, sugarcane (raw sugar) is processed (by grinding) to produce sugarcane juice which is then heated to extract the white table sugar (sucrose). The remaining crystalized sugar particles are further processed to be sold as animal feed. In the wheat markets, wheat seeds are processed (by grinding and sieving) to produce wheat bran and course powder flour (which is further processed and sold as animal feed).<sup>6</sup> In all these markets, a primary commodity input (cocoa bean, raw sugar, wheat) is processed into two main products in fixed proportions. Since the input is a commodity, the processor can use the spot market and the contract market for procurement.

We model the firm's decision problem as a two-stage stochastic recourse problem. In stage 0, the firm decides the quantity  $Q^i \leq \bar{Q}^i$  to reserve from each contract  $i$ , facing input spot price and output demand uncertainties. At stage 1,  $P^S$  and  $\mathbf{D}$  are realized, and the firm decides the volume of input to exercise from each contract ( $z_i^C \leq Q^i$ ) as well as the volume of spot procurement ( $z^S \leq K$ ). Finally, the firm determines the production quantity of each product  $x_j$  subject to its available yield. This decision also implies the firm's salvaging quantity as a semi-processed input for each product, viz.  $a_j(\mathbf{1}'\mathbf{z}^C + z^S) - x_j$ .<sup>7</sup> The objective of the firm is to maximize the expected total profit at stage 0.

We now formulate the firm's decision problem starting from stage 1:

$$\begin{aligned}
\max_{\mathbf{z}^C, z^S, \mathbf{x}} \quad & -\mathbf{b}'\mathbf{z}^C - P^S z^S - \omega(\mathbf{1}'\mathbf{z}^C + z^S) \\
& + \sum_{j=1}^2 [p_j \min(x_j, D_j) - r_j(D_j - x_j)^+ + \underline{s}_j(x_j - D_j)^+ - c_j x_j + \bar{s}_j(a_j(\mathbf{1}'\mathbf{z}^C + z^S) - x_j)] \\
\text{s.t.} \quad & \mathbf{z}^C \leq \mathbf{Q}, \quad z^S \leq K, \quad \mathbf{x} \leq \mathbf{a}(\mathbf{1}'\mathbf{z}^C + z^S) \\
& \mathbf{z}^C \geq \mathbf{0}, \quad z^S \geq 0, \quad \mathbf{x} \geq \mathbf{0}.
\end{aligned} \tag{1}$$

In (1), the first two terms, the total contract exercise price and the spot procurement cost,

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<sup>6</sup>Course powder flour can also be processed into bioethanol but this requires a more sophisticated technology in the wheat mill.

<sup>7</sup>This model implicitly assumes that the processing of the primary input and the production of the outputs are instantaneous such that the processing and production decisions are made after input spot price and output demands are realized. This is a reasonable assumption for the agricultural product markets considered in this paper.

represent the total procurement cost on the spot day. The third term is the processing cost of the firm. The final terms in the objective function denote the total profit from the two product markets. For each product market, the profit is given by the sum of the sales and salvage revenues from the product and salvage revenues from the semi-processed input less of the production cost and penalty cost for the unsatisfied demand. The first constraint ensures that the firm does not process more than the reserved volume for each contract. The second constraint limits the volume of spot procurement. The third constraint ensures that the firm does not produce more than the available yield for each product  $j$ . Let  $\pi(\mathbf{Q}; P^S, \mathbf{D})$  denote the optimal stage 1 profit for a given  $\mathbf{Q}$ .

Anticipating these decisions, at stage 0, the firm solves for the optimal volume of input to reserve from each contract  $i$ ,  $\mathbf{Q}^*$ , to maximize the expected firm profit:

$$V^* = \max_{\mathbf{0} \leq \mathbf{Q} \leq \bar{\mathbf{Q}}} \mathbb{E}[\pi(\mathbf{Q}; \tilde{P}^S, \tilde{\mathbf{D}})] - \beta' \mathbf{Q} \quad (2)$$

where the expectation is taken over  $\tilde{P}^S$  and  $\tilde{\mathbf{D}}$ . We assume that the distributions of  $\tilde{P}^S$  and  $\mathbf{D}$  are statistically independent. In (2), the second term denotes the total contract reservation cost of the firm. The constraint ensures that the firm does not reserve more than the available capacity  $\bar{Q}^i$  of each contract.

### 3 The Optimal Strategy- Stage 1: Spot Market

In this section, we describe the solution for the firm's optimal sourcing and production decisions after the input spot price and output demand uncertainties are realized. All the proofs are relegated to the Appendix.

We index the products such that  $\frac{D_1}{a_1} \leq \frac{D_2}{a_2}$ . In stage 0, the firm reserved  $Q^i$  units of capacity from contract  $i$  for  $i = 1, 2$ . In this stage, besides these two contracts, the firm can also source input from the spot market at the prevailing spot price  $P^S$  up to the spot procurement capacity  $K$ . The two contracts and the spot procurement constitute the firm's *sourcing arrangements*. We index these sourcing arrangements in increasing order of their procurement cost  $b^{[i]}$  at this stage, i.e.  $b^{[1]} \leq b^{[2]} \leq b^{[3]}$ , where  $b^{[i]}$  is either the exercise price (for contract procurement) or the spot price (for spot procurement). For example, for  $b^1 < P^S < b^2$ , we have  $b^{[1]} = b^1, b^{[2]} = P^S, b^{[3]} = b^2$ . Let  $q^{[i]}$  denote the maximum available capacity of the  $[i]^{th}$  sourcing arrangement (either the reserved contract capacity  $Q^{[i]}$  or the spot procurement capacity  $K$ ), and  $z^{[i]}$  be the corresponding processing quantity such that

$z^{[i]} \leq q^{[i]}$ . Let  $\mathbf{b}^{[1]}$ ,  $\mathbf{q}^{[1]}$ ,  $\mathbf{z}^{[1]}$  denote the exercise price, capacity limit and processing quantity vector of the three sourcing arrangements, respectively.

To find the optimal solution for (1), we decompose the problem and first solve for the optimal production decisions  $x_j$  for  $j = 1, 2$ , given the processed input vector  $\mathbf{z}^{[1]}$ :

$$\begin{aligned} \max_{x_j} \quad & p_j \min(x_j, D_j) - r_j(D_j - x_j)^+ + \underline{s}_j(x_j - D_j)^+ - c_j x_j + \bar{s}_j(a_j \mathbf{1}' \mathbf{z}^{[1]} - x_j) \\ \text{s.t.} \quad & 0 \leq x_j \leq a_j \mathbf{1}' \mathbf{z}^{[1]}. \end{aligned} \quad (3)$$

As depicted in the formulation above, the firm decides the allocation of the available yield  $a_j \mathbf{1}' \mathbf{z}^{[1]}$  for product  $j$  between the production ( $x_j$ ) and the semi-processed input salvage market ( $a_j \mathbf{1}' \mathbf{z}^{[1]} - x_j$ ).

**Proposition 1** *For  $j = 1, 2$ , the unique optimal production quantity  $x_j^*$  for a given processed input vector  $\mathbf{z}^{[1]}$  is given by*

$$x_j^* = \begin{cases} 0 & \text{if } c_j + \bar{s}_j \geq p_j + r_j \geq \underline{s}_j \\ \min(a_j \mathbf{1}' \mathbf{z}^{[1]}, D_j) & \text{if } p_j + r_j \geq c_j + \bar{s}_j \geq \underline{s}_j \\ a_j \mathbf{1}' \mathbf{z}^{[1]} & \text{if } p_j + r_j \geq \underline{s}_j \geq c_j + \bar{s}_j. \end{cases} \quad (4)$$

The optimal profit from product  $j$ ,  $\Psi_j(\mathbf{z}^{[1]})$ , is characterized by

$$\Psi_j(\mathbf{z}^{[1]}) = \hat{p}_j \min(a_j \mathbf{1}' \mathbf{z}^{[1]}, D_j) - r_j(D_j - a_j \mathbf{1}' \mathbf{z}^{[1]})^+ + \hat{s}_j(a_j \mathbf{1}' \mathbf{z}^{[1]} - D_j)^+ - c_j a_j \mathbf{1}' \mathbf{z}^{[1]} \quad (5)$$

where  $\hat{p}_j \doteq \max(p_j, c_j + \bar{s}_j - r_j) \geq 0$  is the effective unit sales price, and  $\hat{s}_j \doteq \max(c_j + \bar{s}_j, \underline{s}_j) \geq 0$  is the effective unit salvage rate for product  $j$ .

In determining the optimal production level for each product, the firm considers three different markets: the product sales market, the product salvage market and the semi-processed input salvage market. The optimal production decision in (4) is determined by the ordering of the marginal revenue from each of these markets. The marginal revenue from the product sales market is given by the sum of the product price and the opportunity gain from not incurring the penalty cost for the unsatisfied demand, i.e.  $p_j + r_j$ , whereas the marginal revenue from the product salvage market is given by  $\underline{s}_j$ . The marginal revenue from the semi-processed input salvage market is given by the sum of the unit salvage value of the semi-processed input and the opportunity gain from not incurring the production cost, i.e.  $\bar{s}_j + c_j$ . When the marginal revenue from the semi-processed input market is the

highest, the firm optimally allocates all the available yield  $a_j \mathbf{1}' \mathbf{z}^{[l]}$  to the semi-processed input market and there is no production, i.e.  $x_j^* = 0$ . When the marginal revenue from the product sales market is the highest, the firm optimally produces a positive quantity, and the optimal production level is determined by the ordering of the two salvage markets. In particular, if the marginal revenue from the product salvage market is higher than the marginal revenue from the semi-processed input salvage market, then all the available yield is allocated to production, i.e.  $x_j^* = a_j \mathbf{1}' \mathbf{z}^{[l]}$ . Otherwise, the firm optimally produces a sufficient amount to cover the demand, i.e.  $x_j^* = \min(a_j \mathbf{1}' \mathbf{z}^{[l]}, D_j)$ .

The optimal profit expression in (5) is equivalent to a classical newsvendor profit expression with a production quantity  $a_j \mathbf{1}' \mathbf{z}^{[l]}$  and an adjusted unit product price and unit salvage value. These adjusted parameters capture the three different markets available for the firm. In particular, the effective unit sales price  $\hat{p}_j$  is the maximum unit revenue from the two sales outlets available for the firm, the product sale ( $p_j$ ) and semi-processed input salvaging ( $c_j + \bar{s}_j - r_j$ ). Similarly, the effective unit salvage value  $\hat{s}_j$  is the maximum unit revenue from the two salvage outlets available, product salvaging ( $\underline{s}_j$ ) and semi-processed input salvaging ( $c_j + \bar{s}_j$ ). Let  $\hat{\mathbf{p}}_j \in \mathbb{R}^2$  and  $\hat{\mathbf{s}}_j \in \mathbb{R}^2$  denote the effective sales price vector and the effective salvage vector, respectively. In the rest of the analysis, we will focus on  $\hat{\mathbf{p}}_j$  and  $\hat{\mathbf{s}}_j$  as our revenue parameters.

As the optimal production vector  $(x_1^*, x_2^*)$  is uniquely defined by  $\mathbf{z}^{[l]}$ , we can optimize the stage 1 problem over the processing vector  $\mathbf{z}^{[l]}$ . Since the components  $z^{[i]}$  of  $\mathbf{z}^{[l]}$  are ordered in increasing order of their stage 1 procurement cost, the firm optimally uses the sourcing arrangement  $[i]$  only if the firm has already used all the sourcing arrangements  $[l] < [i]$  up to their capacity limits  $q^{[l]}$ . Using this observation, we can redefine the stage 1 decision problem in (1) as a single variable optimization problem. Let  $z$  denote the total number of input to process and  $\Pi(z)$  denote the stage 1 objective function. Let  $z^*$  denote the optimal solution to this problem. In the optimal solution, for a given  $z^*$ , the optimal allocation  $\mathbf{z}^{[l]*}$  over the three sourcing arrangements is unique and is characterized by  $z^{[i]*} = \max\left(\min\left(z^* - \sum_{k=[0]}^{[i]-1} q^k, q^{[i]}\right), 0\right)$ , where we define  $q^{[0]} \doteq 0$  for notational convenience.

To characterize the stage 1 objective function  $\Pi(z)$ , we first define

$$\Pi^{[i]}(z) \doteq \sum_{j=1}^2 \Psi_j(z) - b^{[i]} z + \sum_{k=[1]}^{[i]} (b^{[i]} - b^k) q^k - \omega z,$$

which is the total profit of the firm when the sourcing arrangement  $[i]$  is used for procure-

ment. The first term denotes the total product market profits, and the last term denotes the processing cost for a given  $z$ . The sum of the second and the third terms denote the total procurement cost of  $z$  at stage 1. This equals the total exercise price  $b^{[i]}z$  minus the total savings from using the cheaper sourcing arrangements  $[l] < [i]$  from their exercise price  $b^{[l]}$  up to their capacity  $q^{[l]}$ .

The stage 1 objective function  $\Pi(z)$  can be obtained from the following two observations. First,  $\Pi^{[i]}(z)$  is *feasible* for  $z \leq \sum_{k=[1]}^{[i]} q^k$  because the firm can only exercise the sourcing arrangement  $[i]$  up to its capacity  $q^{[i]}$ . Second,  $\Pi^{[i]}(z)$  is *relevant* for  $z \geq \sum_{k=[0]}^{[i]-1} q^k$  (where  $[0]$  denotes the dummy sourcing arrangement with  $q^{[0]} = 0$ ) because the firm uses the sourcing arrangement  $[i]$  only when all the cheaper sourcing arrangements are used up to their capacity. Therefore, the stage 1 objective function can be written as

$$\Pi(z) = \Pi^{[i]}(z) \text{ for } \sum_{k=[0]}^{[i]-1} q^k \leq z \leq \sum_{k=[0]}^{[i]} q^k, \quad [i] = [1], [2], [3]. \quad (6)$$

We now characterize the optimal processing decision  $z^*$  that maximizes  $\Pi(z)$ . We first provide additional notation that facilitates the analysis. Let  $[i]^j \in \{[1], [2], [3], [4]\}$  denote the index of the cheapest sourcing arrangement that can satisfy the demand of product  $j$ , i.e. the smallest  $[i]$  for product  $j \in \{1, 2\}$  such that  $\frac{D_j}{a_j} < \sum_{k=[1]}^{[i]} q^k$ . Here,  $[4]$  denotes a dummy sourcing arrangement with  $b^{[4]} \rightarrow \infty$  and  $q^{[4]} \rightarrow \infty$ . Let  $j^{[i]} \in \{0, 1, 2\}$  denote index of the product with the highest demand which can be satisfied by sourcing arrangement  $i$ , i.e. the largest  $j$  for a given  $[i] \in \{[1], [2], [3]\}$  such that  $\frac{D_j}{a_j} < \sum_{k=[1]}^{[i]} q^k$ . Here,  $j^{[i]} = 0$  implies  $\frac{D_1}{a_1} \geq \sum_{k=[1]}^{[3]} q^k$ .

**Proposition 2** *When the elements of  $C \doteq \{0, \frac{D_1}{a_1}, \frac{D_2}{a_2}, \sum_{k=[1]}^{[i]} q^k \quad \forall [i]\}$  are non-identical, the unique optimal processing decision  $z^*$  is characterized by*

$$z^* = \begin{cases} 0 & \text{if } h_3 \leq b^{[1]} \\ \frac{D_j}{a_j} \text{ for } j = 1, 2 & \text{if } h_{(4-j)} \geq b^{[i]^j} \text{ and } h_{(3-j)} < b^{[i]^j} \\ \sum_{k=[1]}^{[i]} q^k \text{ for } [i] = [1], [2], [3] & \text{if } h_{(3-j^{[i]})} \geq b^{[i]} \text{ and } h_{(3-j^{[i]})} < b^{[i]+1} \end{cases} \quad (7)$$

where  $h_1 \leq h_2 \leq h_3$  is defined as

$$h_1 \doteq a_1(\widehat{s}_1 - c_1) + a_2(\widehat{s}_2 - c_2) - \omega, \quad (8)$$

$$h_2 \doteq a_1(\widehat{s}_1 - c_1) + a_2(\widehat{p}_2 + r_2 - c_2) - \omega,$$

$$h_3 \doteq a_1(\widehat{p}_1 + r_1 - c_1) + a_2(\widehat{p}_2 + r_2 - c_2) - \omega.$$

When some of the elements of  $C$  are identical,  $z^*$  is given by

$$z^* = \begin{cases} \frac{D_1}{a_1} = \frac{D_2}{a_2} & \text{if } h_3 \geq b^{[i]^1} \text{ and } h_3 < b^{[i]^2} \\ \sum_{k=[1]}^{[t]} q^k = \sum_{k=[1]}^{[u]} q^k \text{ for largest } [u] \text{ such that } [u] > [t] & \text{if } h_{(3-j^{[t]})} \geq b^{[t]} \text{ and } h_{(3-j^{[u]})} < b^{[u]+1} \\ \frac{D_l}{a_l} = \frac{D_s}{a_s} = \sum_{k=[1]}^{[t]} q^k = \sum_{k=[1]}^{[u]} q^k \text{ for } s \geq l \text{ and } u \geq t & \text{if } h_{(3-j^{[t]})} \geq b^{[t]} \text{ and } h_{(3-s)} < b^{[i]^s}. \end{cases}$$

The stage 1 objective function  $\Pi(z)$  is piecewise linear and concave in  $z$ . Therefore, the optimal solution occurs at the breakpoints  $\{0, \frac{D_1}{a_1}, \frac{D_2}{a_2}, \sum_{k=[1]}^{[i]} q^k \ \forall [i]\}$  that define the form of  $\Pi(z)$  in (6). To demonstrate the intuition, we focus on the case with non-identical breakpoints, and the optimality condition in (7). The firm optimally does not process when the marginal procurement cost of the first unit, i.e. the price  $b^{[1]}$  of the cheapest sourcing arrangement, is larger than the marginal processing revenue  $h_3$ . Since each unit of input generates  $a_j$  units of product  $j$ , and the demand of each product is strictly larger than 0, the marginal processing revenue  $h_3$  is given by the sum of the unit sales revenue from each product market,  $a_j(\hat{p}_j + r_j - c_j)$ , minus the unit processing cost  $\omega$ .

If the firm optimally processes, then the characterization of  $z^*$  depends on the ordering of  $\{\frac{D_1}{a_1}, \frac{D_2}{a_2}, \sum_{k=[1]}^{[i]} q^k \ \forall [i]\}$ . The marginal procurement cost of an additional unit is determined by the price of the relevant sourcing arrangement. The marginal revenue of processing this additional unit is determined by the sales or salvaging options available in each product market: In (8),  $h_1$  denotes the marginal revenue of processing when both products are salvaged,  $h_2$  denotes the marginal revenue of processing when the first product is salvaged and the second product is sold, and  $h_3$  denotes the same when both products are sold.

Let us focus on the example where  $q^{[1]} < \frac{D_1}{a_1} < q^{[1]} + q^{[2]} < \frac{D_2}{a_2} < q^{[1]} + q^{[2]} + q^{[3]}$ , i.e. the demand in the first product market can only be satisfied if the two cheapest sourcing arrangements are used, whereas the demand in the second product market can only be satisfied if all the sourcing arrangements are used. In this case, we have  $[i]^1 = [2]$ ,  $[i]^1 = [3]$  and  $j^{[1]} = 0$ ,  $j^{[2]} = 1$ ,  $j^{[3]} = 2$ . For instance, the firm optimally processes  $z^* = \frac{D_1}{a_1}$  if the unit procurement cost with the second cheapest sourcing arrangement ( $b^{[2]}$ ) is smaller than the marginal revenue of processing when both products are sold ( $h_3$ ), but is larger than the marginal revenue of processing when the first product is salvaged and the second product is sold ( $h_2$ ). It is optimal to process all the available units, i.e.  $z^* = q^{[1]} + q^{[2]} + q^{[3]}$ , if the marginal revenue of processing when both products are salvaged ( $h_1$ ), is larger than the unit procurement cost of the most expensive sourcing arrangement ( $b^{[3]}$ ). The rest of the optimal solution follows a similar structure.

We close this section with three remarks that we will use throughout the paper. First, we will assume that the unit processing cost  $\omega$  is sufficiently low such that  $h_1 \geq 0$ .<sup>8</sup> Second, we will assume  $b^i \in [0, h_3)$  for each contract  $i$ . When  $b^i \geq h_3$ , contract  $i$  is never exercised at stage 1, thus the firm does not use this contract at stage 0. Third, since the demands are stochastic at stage 0,  $h_2$ , the marginal revenue of processing when the first product is salvaged and the second product is sold, takes two different forms depending on the demand realizations. Therefore, when we analyze the optimal contracting decision in the next section, we arbitrarily index the two products and define

$$\begin{aligned} h_2^1 &\doteq a_1(\widehat{s}_1 - c_1) + a_2(\widehat{p}_2 + r_2 - c_2) - \omega, \\ h_2^2 &\doteq a_1(\widehat{p}_1 + r_1 - c_1) + a_2(\widehat{s}_2 - c_2) - \omega. \end{aligned} \tag{9}$$

In (9),  $h_2^1$  denotes the marginal revenue of processing when the first product is salvaged and the second product is sold (and is relevant for  $\frac{D_1}{a_1} \leq \frac{D_2}{a_2}$ ), and  $h_2^2$  denotes the marginal revenue of processing when the first product is sold and the second one is salvaged (and is relevant for  $\frac{D_1}{a_1} \geq \frac{D_2}{a_2}$ ). We will assume  $h_2^1 \leq h_2^2$  which, in turn, implies  $h_1 \leq h_2^1 \leq h_2^2 \leq h_3$ .

## 4 The Optimal Strategy- Stage 0: Contract Market with A Single Contract

At this stage, the firm chooses the contract volume to reserve to maximize its expected profit in the presence of input spot price and output demand uncertainties. In this section, we focus on a single contract version of our model, i.e.  $i = 1$ , with reservation price  $\beta$ , exercise price  $b$ , and maximum capacity  $\bar{Q}$  available for contracting. We analyze the impact of the input spot price uncertainty and the output demand uncertainty on the optimal expected profit and the optimal contract volume of the firm, and provide managerial insights. For these comparative statics, we assume  $\tilde{P}^S$  to follow a normal distribution and  $\tilde{\mathbf{D}}$  to follow a bivariate normal distribution. Our analysis focuses on determining the impact of two factors on these comparative statics results.

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<sup>8</sup>Note that our analysis requires  $h_3 > 0$ ; otherwise the firm cannot generate any revenues at stage 1. When the remaining marginal revenue terms ( $h_1, h_2^1, h_2^2$ ) take negative values, all of our analytical results continue to hold but the discussion of these results are more involved. To avoid unnecessary complications, we assume  $h_1 \geq 0$ , i.e. the marginal revenue of processing is always non-negative.

First, we analyze how these results are impacted by the firm's spot procurement access. To this end, we focus on two special cases of our model: *i*) no spot procurement access case, that is obtained by setting  $K = 0$ ; and *ii*) the unlimited spot procurement access case, that is obtained by setting  $K \rightarrow \infty$ . We investigate how the impact of demand uncertainty on the optimal contract volume and the expected profit of the firm change with the spot procurement access by comparing the  $K = 0$  and  $K \rightarrow \infty$  cases.

Second, we analyze how these comparative static results are impacted by the contract parameters. To this end, we investigate the impact of the exercise price  $b$  of the contract on our results. If the impact of  $b$  on a comparative statics result is not reported, this implies that the result holds for any  $b \in [0, h_3)$ . As a special case of this analysis, comparing the comparative statics results with  $b = 0$  and  $b > 0$  illustrates the differences between the forward and the option contracts.

In §4.1, we analyze the no spot procurement access case. §4.2 analyzes the unlimited spot procurement access case. We summarize our findings from these two cases in §4.3 and provide our managerial insights.

#### 4.1 No Spot Procurement Access ( $K = 0$ )

In the absence of spot procurement, the optimal contract volume  $Q^*$  solves  $\max_{0 \leq Q \leq \bar{Q}} V(Q) = \mathbb{E}[\pi(Q; \tilde{\mathbf{D}})] - \beta Q$  where  $\pi(Q; P^S, \mathbf{D})$  denotes the optimal stage 1 profit for a given  $Q$ .

**Proposition 3** *The expected profit of the firm  $V(Q)$  is concave in  $Q$ . The optimal contract volume  $Q^*$  is characterized by the following first-order condition:*

$$\begin{aligned} \frac{\partial V}{\partial Q} = & -\beta - b + h_1 + (h_2^1 - h_1)P\left(\frac{D_2}{a_2} \geq Q\right) + (h_2^2 - h_1)P\left(\frac{D_1}{a_1} \geq Q\right) \\ & + (b - h_1)^+ P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} \leq Q\right) + (b - h_2^1)^+ P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} > Q\right) + (b - h_2^2)^+ P\left(\frac{D_1}{a_1} > Q, \frac{D_2}{a_2} \leq Q\right). \end{aligned} \quad (10)$$

where  $h_1 \leq h_2^1 \leq h_2^2 \leq h_3$  are as given in (8) and (9). We have  $Q^* = 0$  if  $\frac{\partial V}{\partial Q}\Big|_{0^+} \leq 0$ ,  $Q^* = \bar{Q}$  if  $\frac{\partial V}{\partial Q}\Big|_{\bar{Q}^-} \geq 0$ ; otherwise it is the solution to  $\frac{\partial V}{\partial Q} = 0$ .

In (10), the first line denotes the expected marginal profit of contracting an additional unit when that unit is always exercised. The marginal procurement cost is the sum of the reservation price and the exercise price, i.e.  $\beta + b$ . In the worst case scenario, the firm can salvage the two products to generate a marginal revenue of  $h_1$ . When the demand of product  $j$  is sufficiently high, i.e.  $\frac{D_j}{a_j} \geq Q$ , the firm can sell product  $j$ . In this case, the firm generates an additional sales margin over the salvage value of product  $j$  that is given by  $h_2^1 - h_1$  for the first product and  $h_2^2 - h_1$  for the second product.

The second line in (10) captures the expected marginal profit of contracting that comes from the contingent exercise decision after the demands are realized. For  $b \leq h_1$ , since the firm profitably salvages both products, the additional contracted unit is always exercised, and there is no value of contingent exercise decision. For  $b > h_1$ , the exercise decision depends on the demand realizations. For example, when  $h_1 < b \leq h_2^1$ , the firm optimally exercises the additional contracted unit only if a sale is generated in at least one of the product markets. Therefore, when the product demands are sufficiently low, i.e.  $\frac{D_1}{a_1} \leq Q$  and  $\frac{D_2}{a_2} \leq Q$ , this additional unit is not exercised. The expressions in the first line of (10) are characterized by assuming that this additional unit is exercised ( $b$  is deducted) and both products are salvaged ( $h_1$  is generated) when  $\frac{D_1}{a_1} \leq Q$  and  $\frac{D_2}{a_2} \leq Q$ . Since the firm optimally does not exercise this unit under these demand realizations, there is an additional marginal revenue of  $(b - h_1)$ . The interpretation is that when the contracted unit is not exercised, a marginal revenue of  $b$  is generated, i.e. the firm receives back the exercise price  $b$ .

We note an interesting similarity between the optimality condition in (10) and the optimal ordering decision in a traditional single-input, single-output newsvendor problem, which neither has the contingent exercise decision, i.e.  $b = 0$ , nor the fixed proportions technology. If we assume symmetric bivariate distribution of  $\left(\frac{D_1}{a_1}, \frac{D_2}{a_2}\right)$  and  $b \leq h_1$ , then  $Q^*$  is characterized by

$$P\left(\frac{D_j}{a_j} \leq Q^*\right) = \frac{\sum_{i=1}^2 a_i u_i}{\sum_{i=1}^2 a_i (u_i + o_i)} \quad (11)$$

where  $u_j \doteq \widehat{p}_j + r_j - c_j - \omega - b - \beta$  and  $o_j \doteq -\widehat{s}_j + c_j + \omega + b + \beta$  are the unit underage and overage costs for product  $j$  respectively. When the order quantity is lower than the product  $j$  demand, the firm misses the sales revenue  $\widehat{p}_j$ , incurs a penalty cost of  $r_j$ , but avoids the procurement ( $\beta + b$ ), processing ( $\omega$ ) and production ( $c_j$ ) costs. This explains the unit underage cost  $u_j$ . When the order quantity is higher than the product  $j$  demand, the firm incurs the procurement, processing and production costs but recoups some of this cost by salvaging the product. This explains the unit overage cost  $o_j$ . The optimality condition in (11) is the critical fractile solution of the traditional single-input, single-output newsvendor problem with the *effective underage* and *effective overage* costs that arise from the fixed proportions technology. As we discuss next, some of the comparative static results for our model parallel the ones for the traditional newsvendor problem due to this inherent similarity.

**Proposition 4** *The optimal contract volume  $Q^*$  increases in the mean demand  $\mu_j$ ; decreases in the demand variability  $\sigma_j$  if  $Q^* < \frac{\mu_j}{a_j}$ , and increases otherwise. It strictly increases in the demand correlation  $\rho$  for  $b > h_1$ ; otherwise it is independent of  $\rho$ .*

An increase in the mean demand  $\mu_j$  increases the probability of sales in product market  $j$ , thus increasing the marginal revenue of processing. Therefore,  $Q^*$  increases. The impact of product  $j$ 's demand variability  $\sigma_j$  on  $Q^*$  is identical to the traditional single-input, single-output newsvendor problem. A higher demand variability decreases  $Q^*$  when the mean demand  $\mu_j$  is high. This is because the expected marginal revenue of contracting is negatively affected by the low demand realizations as fewer products can be sold; at the same time, the high demand realizations are not fully exploited as the sales are capped with the input availability. When  $\mu_j$  is high, the negative impact of the low demand realizations outweighs the positive impact of the high demand realizations, and the expected marginal revenue decreases. Therefore,  $Q^*$  decreases.

On the impact of the demand correlation  $\rho$ , it follows from (10) that the optimal contract volume is independent of  $\rho$  for  $b \leq h_1$ . This is because the contract is always exercised, and the expected marginal revenue of contracting depends on the marginal distribution of demand (which is independent of  $\rho$ ). For  $b \in (h_1, h_3]$ , the optimal contract volume  $Q^*$  is impacted by  $\rho$  as the expected marginal profit from the contingent exercise decision, the second line in (10), is impacted by  $\rho$ . To explain this, let us consider the case where  $h_1 < b \leq h_2^1$ . In this case, the firm optimally does not exercise the contract if both products are salvaged. Therefore, the expected marginal profit of contracting from contingent exercise decision is given by  $(b - h_1)P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} \leq Q\right)$ . A higher demand correlation  $\rho$  increases the joint probability of observing low demand in each market (which follows from Slepian's inequality). Therefore,  $Q^*$  increases.

We next analyze the impact of the demand uncertainty on the optimal expected profit.

**Proposition 5** *The optimal expected profit  $V^*$  increases in the mean demand  $\mu_j$  when  $r_1 = r_2 = 0$ ; decreases in the demand variability  $\sigma$  for  $\sigma_1 = \sigma_2 = \sigma$ ; and increases in the demand correlation  $\rho$ .*

An increase in the mean demand  $\mu_j$  of product  $j$  increases the optimal expected profit of the firm in the absence of a penalty cost for the unsatisfied demand. This is because the probability of a product sale increases, increasing the value of processing. An increase in

the demand variability  $\sigma$  decreases the optimal expected profit of the firm. This is because the firm is negatively affected by the low demand realizations as fewer products can be sold; at the same time, it cannot fully benefit from the high demand realizations as the sales are capped by the contract volume. An increase in the demand correlation  $\rho$  increases the optimal expected profit as the firm makes better use of its production options in both markets: With a low demand correlation, the demand for one of the products is high when the other one is low. In this case, the firm can only sell one of the products and salvage the other; therefore, there is an imbalance in the product markets. With a higher demand correlation, demands for both products are more aligned, and this imbalance decreases.

We note here that the impact of the demand correlation  $\rho$  is a result of the fixed proportions technology. To demonstrate this, let us consider the case where product  $j$  uses  $\frac{1}{a_j}$  units of the input, but there is no fixed proportions technology. This setting is similar to the single-input, two-output newsvendor network model of Van Mieghem and Rudi (2002) that captures the component commonality (or flexible capacity). In this setting, to produce one unit of each product,  $\frac{1}{a_1} + \frac{1}{a_2}$  units of input should be used. There is a capacity-pooling benefit of the common input, and the expected profit of the firm *decreases* in the demand correlation as demonstrated by Van Mieghem and Rudi (2002). In our setting,  $\max\left(\frac{1}{a_1}, \frac{1}{a_2}\right)$  units of input are used to produce one unit of each product. There is a cost-pooling benefit of the common input due to fixed proportions technology and the expected profit of the firm *increases* in the demand correlation.

## 4.2 Unlimited Spot Procurement Access ( $K \rightarrow \infty$ )

With unlimited spot procurement access, the optimal contract volume  $Q^*$  solves  $\max_{0 \leq Q \leq \bar{Q}} V(Q) = \mathbb{E}[\pi(Q; \tilde{P}^S, \tilde{\mathbf{D}})] - \beta Q$  where  $\pi(Q; P^S, \mathbf{D})$  denotes the optimal stage 1 profit for a given  $Q$ .

**Proposition 6** *The expected profit of the firm  $V(Q)$  is concave in  $Q$ . The optimal contract volume  $Q^*$  is characterized by the following first-order condition:*

$$\begin{aligned} \frac{\partial V}{\partial Q} = & -\beta - b + P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} \leq Q\right) \max(b, h_1) + P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} > Q\right) G(\max(b, h_1), \max(b, h_2^1)) \\ & + P\left(\frac{D_1}{a_1} > Q, \frac{D_2}{a_2} \leq Q\right) G(\max(b, h_1), \max(b, h_2^2)) + P\left(\frac{D_1}{a_1} > Q, \frac{D_2}{a_2} > Q\right) G(\max(b, h_1), h_3) \end{aligned} \quad (12)$$

where  $G(\lambda_0, \lambda_1) \doteq \mathbb{E}[\min(\max(\tilde{P}^S, t_0), \lambda_1)]$  for  $\lambda_0 \leq \lambda_1$ , and  $h_1 \leq h_2^1 \leq h_2^2 \leq h_3$  are as given in (8) and (9). We have  $Q^* = 0$  if  $\frac{\partial V}{\partial Q}\Big|_{0^+} \leq 0$ ,  $Q^* = \bar{Q}$  if  $\frac{\partial V}{\partial Q}\Big|_{\bar{Q}^-} \geq 0$ ; otherwise it is the solution to  $\frac{\partial V}{\partial Q} = 0$ .

The first term in (12),  $\beta + b$ , is the marginal contract procurement cost, whereas the remaining terms characterize the expected value of the marginal revenue of an additional unit of contracted volume. At stage 1, the marginal revenue takes different forms as it depends on the demand realizations as well as the spot procurement option of the firm. To delineate the intuition behind the marginal revenue expressions in (12), we will focus on the first two terms in (12).

First, let us consider the case where both product demands are low at stage 1 such that  $\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} \leq Q$ . In this case, the firm salvages both products and generates a marginal revenue of  $h_1$  if  $b \leq h_1$ ; otherwise the additional contracted unit is not exercised. Therefore, the marginal revenue at stage 1 is given by  $\max(b, h_1)$ . This marginal revenue, multiplied by the probability of its occurrence, i.e.  $P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} \leq Q\right)$ , is part of the expected marginal revenue expression in (12). We note here that since the additional contracted unit can always be salvaged in the two product markets, the spot procurement option does not have an impact on the marginal revenue at stage 1 in this case.

Second, consider the case where  $\frac{D_1}{a_1} \leq Q$  and  $Q < \frac{D_2}{a_2}$  at stage 1. To delineate the intuition, we first focus on the case with  $b = 0$ , i.e. there is no value of contingent exercise at stage 1. At these demand realizations, the additional contracted unit can be salvaged in the first product market regardless of the spot procurement option. In the second product market, the spot procurement option plays a key role. In the absence of spot procurement, the contracted unit can be sold in the second market (as  $Q < \frac{D_2}{a_2}$ ). If the firm uses spot procurement then the contracted unit can only be salvaged in the second market. Therefore, the marginal revenue at stage 1 is determined by the relative attractiveness of the spot procurement option. If the spot price  $P^S$  is greater than  $h_2^1$  (the marginal revenue of processing when the first product is salvaged and the second product is sold) then spot procurement is not cost effective. At these spot price realizations, the marginal revenue of the additional contracted unit is  $h_2^1$ . If  $P^S$  is smaller than  $h_2^1$ , then spot procurement is cost effective. Therefore, at these spot price realizations, the marginal revenue is lower than  $h_2^1$ . In the worst case scenario, the firm can always salvage both products, and generate a marginal revenue of  $h_1$ . Since the contracted unit is used before any spot procurement, for  $P^S \in (h_1, h_2^1]$ , the marginal revenue of the additional unit of contract is the opportunity gain of not procuring from the spot market,  $P^S$ . In summary, with  $b = 0$ , the marginal

revenue at stage 1 is given by

$$\begin{cases} h_1 & \text{if } h_1 \geq P^S, \\ P^S & \text{if } h_1 < P^S \leq h_2^1, \\ h_2^1 & \text{if } h_2^1 > P^S. \end{cases} \quad (13)$$

We define  $G(h_1, h_2^1) \doteq \mathbb{E}[\min(\max(\tilde{P}^S, h_1), h_2^1)]$  as the expected value of (13) where the expectation is taken with respect to the input spot price uncertainty.  $G(h_1, h_2^1)$ , multiplied by the probability of its occurrence, i.e.  $P\left(\frac{D_1}{a_1} \leq Q, Q < \frac{D_2}{a_2} \leq Q + K\right)$ , is part of the expected marginal revenue expression in (12). With  $b > 0$ , the firm receives back  $b$  when the additional unit is optimally not exercised, thus  $h_1$  and  $h_2^1$  are replaced with  $\max(b, h_1)$  and  $\max(b, h_2^1)$  respectively in the  $G(\cdot)$  expression.

The rest of the expected marginal revenue expression in (12) follows a similar structure. The marginal revenue at stage 1 depends on the spot procurement option of the firm, and is characterized by a processing window defined for  $\lambda_0 \leq \lambda_1$ . In particular, when the spot price is inside this window, i.e.  $P^S \in [\lambda_0, \lambda_1]$ , the marginal revenue at stage 1 is characterized by the opportunity gain from not using spot procurement; and outside this window, it is characterized by  $\lambda_1$  when the spot price is higher, and  $\lambda_0$  when the spot price is lower. Here,  $\lambda_1$  denotes the marginal revenue of processing if there is no spot procurement; and  $\lambda_0$  denotes the marginal revenue of processing when the spot procurement is used up to its maximum capacity  $K$ . The parameters of the processing window  $(\lambda_0, \lambda_1)$  are determined by the sales or salvaging options available in each market (which are contingent on the demand realizations  $(D_1, D_2)$ ), as well as the value of the contingent exercise decision of the contracted unit at stage 1 (which depends on the exercise price  $b$ ). Therefore, the expected marginal revenue at the contracting stage is given by sum of the product of  $G(\lambda_0, \lambda_1)$ , i.e. the expected value of the processing window, with the probability of its occurrence, i.e. the probability of demand realizations inducing  $\lambda_0$  and  $\lambda_1$ .

We first analyze the impact of input spot price uncertainty on the optimal contract volume.

**Proposition 7** *The optimal contract volume  $Q^*$  increases in the mean spot price  $\mu_S$ . There exists a unique  $\hat{\mu}_S(b) \in \left[\frac{\max(b, h_1) + \max(b, h_2^1)}{2}, \frac{\max(b, h_2^1) + h_3}{2}\right]$  such that  $Q^*$  increases in the spot price variability  $\sigma_S$  when  $\mu_S < \hat{\mu}_S(b)$  (and decreases otherwise).  $\hat{\mu}_S(b)$  is increasing in  $b$ .*

The impact of the input spot price uncertainty on the optimal contract volume is determined by its impact on the  $G(\lambda_0, \lambda_1)$  terms in (12). An increase in the mean spot price  $\mu_S$  increases  $G(\lambda_0, \lambda_1)$ , and  $Q^*$  increases. The intuition is that as the mean spot price increases, spot procurement cost increases. Therefore, the firm relies less on spot procurement and more on contract procurement. With an increase in the spot price variability  $\sigma_S$ ,  $G(\lambda_0, \lambda_1)$  increases if  $\mu_S \leq \frac{\lambda_0 + \lambda_1}{2}$  (and decreases otherwise). This is because when the mean spot price is sufficiently low, with a higher  $\sigma_S$ , the contract benefits from high  $P^S$  realizations (when the marginal revenue at stage 1 is characterized by the opportunity gain from not using spot procurement) whereas it is not negatively affected from low  $P^S$  realizations as much due to the downside protection  $\lambda_0$  in the processing window. Considering the impact of  $\sigma_S$  on all  $G(\lambda_0, \lambda_1)$  terms in (12), there exists a unique  $\hat{\mu}_S(b)$  such that when  $\mu_S < \hat{\mu}_S(b)$ , the expected marginal revenue of contracting increases, and  $Q^*$  increases (and decreases otherwise). As the exercise price  $b$  of the contract increases,  $(\lambda_0, \lambda_1)$  that define the processing window increase, thus  $G(\lambda_0, \lambda_1)$  increases. Therefore,  $\hat{\mu}_S(b)$  also increases in  $b$ .

The impact of the output demand uncertainty on the optimal contract volume is identical to the no spot procurement access case as given in Proposition 4 except one modification:

**Proposition 8** *The optimal contract volume  $Q^*$  increases in the mean demand  $\mu_j$ , and decreases in the demand variability  $\sigma_j$  if  $Q^* < \frac{\mu_j}{a_j}$  (and increases otherwise). There exists a unique  $\bar{b} \in [h_1, h_2^1]$  such that  $Q^*$  decreases in the demand correlation  $\rho$  when  $b \leq \bar{b}$  (and increases otherwise).*

The impact of the demand correlation  $\rho$  is different from the no spot procurement access case. To delineate the intuition, we will focus on two examples. First, consider the case  $b \leq h_1$ , i.e. the contracted input is always processed as the exercise price is lower than the marginal revenue of processing when both products are salvaged. With a higher  $\rho$ , the probability of observing high or low demand in both markets, the first and fourth probability terms in (12), increase whereas the probability of observing a high demand in one market, and a low demand in the other market, the second and the third probability terms in (12), decrease. These probability terms are impacted by  $\rho$  in the same magnitude, thus the total impact is determined by the expected value of processing window, i.e.  $G(\lambda_0, \lambda_1)$ , under these four demand scenarios. It follows that  $G(h_1, h_1) - G(h_1, h_2^1) - G(h_1, h_2^2) + G(h_1, h_3) < 0$ , and  $Q^*$  decreases in  $\rho$ . In the absence of spot procurement, the marginal revenue at stage 1

under these four demand scenarios is given by  $\lambda_1$  of  $G(\lambda_0, \lambda_1)$ . We have  $h_1 - h_2^1 - h_2^2 + h_3 = 0$ , i.e. the impact of  $\rho$  under these four demand scenarios cancel each other and thus,  $Q^*$  does not change in  $\rho$ . Second, consider the case  $h_2^2 < b < h_3$ , i.e. it is optimal to process the contracted input only if both products can be sold in their markets. In this case, the optimality condition in (12) can be rewritten as  $-\beta + P\left(\frac{D_1}{a_1} > Q, \frac{D_2}{a_2} > Q\right)(G(b, h_3) - b)$ . The joint probability term is increasing in  $\rho$  and thus,  $Q^*$  increases in  $\rho$ . This parallels the impact of  $\rho$  in the absence of spot procurement.

We next analyze the impact of the input spot price and the output demand uncertainties on the optimal expected profit of the firm.

**Proposition 9** *The optimal expected profit  $V^*$  increases in the mean demand  $\mu_j$  when  $r_1 = r_2 = 0$ ; decreases in the demand variability  $\sigma$  for  $\sigma_1 = \sigma_2 = \sigma$ ; increases in the demand correlation  $\rho$ ; decreases in the mean spot price  $\mu_S$ , and increases in the spot price variability  $\sigma_S$ .*

An increase in the mean spot price  $\mu_S$  increases the expected spot procurement cost and thus, decreases the optimal expected profit of the firm. An increase in the spot price variability  $\sigma_S$  increases the optimal expected profit of the firm. This is because the firm optimally procures from the spot market when the spot price is sufficiently low, and thus, benefits from a lower expected spot procurement cost with an increase in  $\sigma_S$ . The impact of the demand uncertainty on the expected optimal profit and its intuition are identical to the no spot procurement access case.

### 4.3 Discussion

In this section, we present managerial insights from the comparative statics results in §4.1 and §4.2, as summarized in Table 1. Our results on the optimal expected firm profit show that, paralleling the single-input, multiple-output newsvendor without fixed proportions technology, the firm benefits from a lower demand variability and a higher expected demand regardless of its contract structure. In contrast to the single-input, multiple-output newsvendor without fixed proportions technology, the firm benefits from a higher demand correlation. Interestingly, these results hold true regardless of the firm's contract structure or its spot procurement access. With spot procurement access, the firm benefits from a lower expected spot price and a higher spot price variability regardless of its contract structure.

An Increase In	No Spot Procurement Access ( $K = 0$ )		Unlimited Spot Procurement Access ( $K \rightarrow \infty$ )	
	Impact on $Q^*$	Impact on $V^*$	Impact on $Q^*$	Impact on $V^*$
Demand Correlation ( $\rho$ )	$\uparrow^{\S}$	$\uparrow$	$\downarrow$ if $b \leq \hat{b}^\dagger$ $\uparrow$ if $b > \hat{b}$	$\uparrow$
Demand Variability ( $\sigma_j$ )	$\uparrow$ if $Q^* \geq \frac{\mu_j}{a_j}$ $\downarrow$ if $Q^* < \frac{\mu_j}{a_j}$	$\downarrow$ if $\sigma_1 = \sigma_2 = \sigma$	$\uparrow$ if $Q^* \geq \frac{\mu_j}{a_j}$ $\downarrow$ if $Q^* < \frac{\mu_j}{a_j}$	$\downarrow$ if $\sigma_1 = \sigma_2 = \sigma$
Mean Demand ( $\mu_j$ )	$\uparrow$	$\uparrow$ if $r_1 = r_2 = 0$	$\uparrow$	$\uparrow$ if $r_1 = r_2 = 0$
Spot Price Variability ( $\sigma_S$ )	NA	NA	$\uparrow \mu_S < \hat{\mu}_S(b)^\ddagger$ $\downarrow \mu_S \geq \hat{\mu}_S(b)$	$\uparrow$
Mean Spot Price ( $\mu_S$ )	NA	NA	$\uparrow$	$\downarrow$

Table 1: The impact of the input spot price uncertainty and the output demand uncertainty on the expected optimal profit ( $V^*$ ) and the optimal contract volume ( $Q^*$ ) of the firm, and the differences in these impacts with respect to the firm’s contract structure (parameterized by its exercise price  $b$ ) and the firm’s spot procurement access (no-access versus unlimited access). These results are proven analytically by assuming a normal distribution of spot price and bivariate normal distribution of demand. NA denotes “Not Applicable”,  $\uparrow$  denotes an increase and  $\downarrow$  denotes a decrease.  $^{\S} Q^*$  strictly increases for  $b > h_1$ , and is constant for  $b \leq h_1$ ;  $^\dagger \hat{b} \in [h_1, h_2^1]$ ;  $^\ddagger \hat{\mu}_S(b)$  is strictly increasing in  $b$  for  $b > h_1$  and is constant in  $b$  for  $b \leq h_1$ .

Our analysis of the optimal contract volume of the firm suggests some rules of thumb for supply management. On the impact of input spot price uncertainty, with an increase in expected spot price, the firm should decrease its contract volume. This result is independent of the firm’s contract structure. Interestingly, the impact of spot price variability on the optimal contract volume depends on the firm’s contract structure. With an increase in spot price variability, the firm should increase (decrease) the contract volume when the expected input spot price is lower (higher) than a threshold; and this threshold increases in the exercise price of the contract. This finding highlights that using a forward or an option contract may lead to diametrically opposed operational choices in response to a change in input spot price variability.

On the impact of demand uncertainty, with an increase in expected demand, the firm

should increase its contract volume regardless of its contract structure or its spot procurement access. The impact of demand correlation on the optimal contract volume is more subtle and depends on the interplay between the contract structure and the spot procurement access. With spot procurement access, with an increase in demand correlation, the firm should decrease its contract volume when the contract has a low execution price (for example, a forward contract); otherwise the firm should increase its contract volume. In the absence of spot procurement, with an increase in demand correlation, the firm should always increase its contract volume when the optimal contract volume is affected by the demand correlation. The firm's optimal contract volume is not affected by the demand correlation only when the firm uses a contract with a low execution price (for example, a forward contract). With an increase in demand variability, the firm should increase (decrease) its contract volume when the expected demand is sufficiently low (high). In contrast to the impact of demand correlation, the impact of demand variability is independent of the firm's contract structure and its spot procurement access.

## 5 The Optimal Strategy- Stage 0: Contract Market with Two Contracts

In §4, we characterized the optimal contracting decision of the firm with a single contract. In this section, we characterize the same when there are two contracts available, i.e.  $i = 1, 2$ , and the firm has unlimited spot procurement access, i.e.  $K \rightarrow \infty$ . The two contracts are characterized by their reservation price, execution price and the maximum capacity  $(\beta^i, b^i, \bar{Q}^i)$ , and are indexed such that  $b^1 \leq b^2$ . The optimal contracting decision  $(Q^{1*}, Q^{2*})$  solves  $\max V(Q^1, Q^2) = \mathbb{E}[\pi(Q^1, Q^2; \tilde{P}^S, \tilde{\mathbf{D}})] - \beta^1 Q^1 - \beta^2 Q^2$  with respect to the constraint set  $\{0 \leq Q^i \leq \bar{Q}^i \text{ for } i = 1, 2\}$ .

We first solve for the deterministic demand benchmark case where the two products are indexed such that  $\frac{D_1}{a_1} \leq \frac{D_2}{a_2}$ . In this case, the marginal revenue of processing under different demand scenarios are characterized by  $h_1 \leq h_2 \leq h_3$  as given in (8).

**Proposition 10** *With deterministic demand, the optimal contract volume is characterized*

by

$$Q^{[1]*} = \begin{cases} 0 & \text{if } \beta^{[1]} + b^{[1]} \geq G(\max(h_1, b^{[1]}), h_3) , \\ \min(\frac{D_1}{a_1}, \bar{Q}^{[1]}) & \text{if } \begin{aligned} & G(\max(h_1, b^{[1]}), h_3) > \beta^{[1]} + b^{[1]} \\ & \geq G(\max(h_1, b^{[1]}), \max(h_2, b^{[1]})) \end{aligned} , \\ \min(\frac{D_2}{a_2}, \bar{Q}^{[1]}) & \text{if } \begin{aligned} & G(\max(h_1, b^{[1]}), \max(h_2, b^{[1]})) > \beta^{[1]} + b^{[1]} \\ & \geq G(\max(h_1, b^{[1]}), \max(h_1, b^{[1]})) \end{aligned} , \\ \bar{Q}^{[1]} & \text{if } \max(h_1, b^{[1]}) > \beta^{[1]} + b^{[1]} , \end{cases}$$

$$Q^{[2]*} = \begin{cases} 0 & \text{if } \beta^{[2]} + b^{[2]} \geq G(\max(h_1, b^{[2]}), h_3) , \\ \min((\frac{D_1}{a_1} - Q^{[1]*})^+, \bar{Q}^{[2]}) & \text{if } \begin{aligned} & G(\max(h_1, b^{[2]}), h_3) > \beta^{[2]} + b^{[2]} \\ & \geq G(\max(h_1, b^{[2]}), \max(h_2, b^{[2]})) \end{aligned} , \\ \min((\frac{D_2}{a_2} - Q^{[1]*})^+, \bar{Q}^{[2]}) & \text{if } \begin{aligned} & G(\max(h_1, b^{[2]}), \max(h_2, b^{[2]})) > \beta^{[2]} + b^{[2]} \\ & \geq G(\max(h_1, b^{[2]}), \max(h_1, b^{[2]})) \end{aligned} , \\ \bar{Q}^{[2]} & \text{if } \max(h_1, b^{[2]}) > \beta^{[2]} + b^{[2]} , \end{cases}$$

where [1] denotes the contract with the higher value of  $-\beta^i - b^i + G(\max(h_1, b^i), h_3)$ , and [2] denotes the other.

The optimal contracting decision follows a greedy policy: The contracts are ranked in decreasing order of an index, and the second contract is used only after the capacity of the first contract is exhausted. The index,  $-\beta^i - b^i + G(\max(h_1, b^i), h_3)$ , is the marginal profit of contracting the first unit of input with contract  $i$ . It is the expected value of the processing window for the first contracted unit minus the marginal procurement cost  $\beta^i + b^i$ . The processing window for the first unit is characterized by  $\max(h_1, b^i)$  (in the worst case scenario, the first unit is either processed and salvaged in both product markets or not exercised) and  $h_3$  (in the best case scenario, the first unit is processed and sold in both product markets). We note here that if the firm uses a single-sourcing policy, i.e. if the firm needs to choose only one of the contracts, it follows from Proposition 10 that the preferred contract is the one with the higher value of this index when the maximum capacity  $\bar{Q}^i$  of the preferred contract is not less than the other.<sup>9</sup>

When the demands are stochastic, the optimal contracting decision is more complex.

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<sup>9</sup>When the maximum capacity of the higher-ranked contract is lower, the preferred contract can only be determined by comparing the optimal expected profit of the firm with each contract. This is because the higher-ranked contract has a higher marginal profit, but it also has a lower availability.

**Proposition 11** *With stochastic demand, the optimal contract volume is characterized by*

$$Q^{1*} = \begin{cases} 0 & \text{if } H^1(0, 0) \leq 0 \\ \arg\{H^1(Q_1^*, 0) = 0\} & \text{if } H^1(0, 0) > 0 \text{ and } H^1(\hat{Q}^1, 0) \leq 0 \\ \hat{Q}^1 & \text{if } H^1(\hat{Q}^1, 0) > 0 \text{ and } H^1(\hat{Q}^1 + \bar{Q}^2, \bar{Q}^2) \leq 0 \\ \arg\{H^1(\bar{Q}^2 + Q^{1*}, \bar{Q}^2) = 0\} & \text{if } H^1(\hat{Q}^1 + \bar{Q}^2, \bar{Q}^2) > 0 \text{ and } H^1(\bar{Q}^1 + \bar{Q}^2, \bar{Q}^2) \leq 0 \\ \bar{Q}^1 & \text{if } H^1(\bar{Q}^1 + \bar{Q}^2, \bar{Q}^2) > 0 \end{cases} \quad (14)$$

$$Q^{2*} = \begin{cases} 0 & \text{if } H^2(\hat{Q}^1, 0) \leq 0 \\ \arg\{H^2(\hat{Q}^1 + Q^{2*}, 0) = 0\} & \text{if } H^2(\hat{Q}^1, 0) > 0 \text{ and } H^2(\hat{Q}^1 + \bar{Q}^2, 0) \leq 0 \\ \bar{Q}^2 & \text{if } H^2(\hat{Q}^1 + \bar{Q}^2, 0) > 0 \end{cases} \quad (15)$$

where  $\hat{Q}^1 = 0$  if  $H^1(0, 0) \leq H^2(0, 0)$ ,  $\hat{Q}^1 = \bar{Q}^1$  if  $H^1(\bar{Q}^1, 0) \geq H^2(\bar{Q}^1, 0)$ , otherwise  $\hat{Q}^1$  is the unique solution to  $H^1(\hat{Q}^1, 0) = H^2(\hat{Q}^1, 0)$ , and

$$\begin{aligned} H^i(x, y) &= -\beta^i - b^i & (16) \\ &+ P\left(\frac{D_1}{a_1} \leq x - y, \frac{D_2}{a_2} \leq x - y\right) \max(\min(h_1, b^2), b^1) \\ &+ P\left(\frac{D_1}{a_1} \leq x - y, \frac{D_2}{a_2} > x - y\right) G(\max(b^1, h_1), \max(\min(b^2, h_2^1), b^1)) \\ &+ P\left(\frac{D_1}{a_1} > x - y, \frac{D_2}{a_2} \leq x - y\right) G(\max(b^1, h_1), \max(\min(b^2, h_2^2), b^1)) \\ &+ P\left(\frac{D_1}{a_1} > x - y, \frac{D_2}{a_2} > x - y\right) G(\max(b^1, h_1), b^2) \\ &+ P\left(\frac{D_1}{a_1} \leq x, \frac{D_2}{a_2} \leq x\right) [\max(h_1, b^i) - \max(\min(h_1, b^2), b^1)] \\ &+ P\left(\frac{D_1}{a_1} \leq x, \frac{D_2}{a_2} > x\right) [G(\max(b^i, h_1), \max(b^i, h_2^1)) - G(\max(b^1, h_1), \max(\min(b^2, h_2^1), b^1))] \\ &+ P\left(\frac{D_1}{a_1} > x, \frac{D_2}{a_2} \leq x\right) [G(\max(b^i, h_1), \max(b^i, h_2^2)) - G(\max(b^1, h_1), \max(\min(b^2, h_2^2), b^1))] \\ &+ P\left(\frac{D_1}{a_1} > x, \frac{D_2}{a_2} > x\right) [G(\max(b^i, h_1), h_3) - G(\max(b^1, h_1), b^2)]. \end{aligned}$$

The expected profit of the firm is jointly concave in  $(Q^1, Q^2)$ , and the optimal contracting decision can be characterized by two optimality conditions, one for each contract. Proposition 11 illustrates the optimal solution  $(Q^{1*}, Q^{2*})$  by using a single function  $H^i(x, y)$ . In the optimal solution, the firm starts sourcing from contract 1 up to a level  $\hat{Q}^1$ , then switches to contract 2 and exhausts its capacity  $\bar{Q}^2$ , and then continues to source from contract 1. The switching point  $\hat{Q}^1$  depends on the cost and revenue parameters. If the marginal profit of sourcing the first unit of input from contract 1 is lower than the marginal profit of sourcing

the same from contract 2, i.e.  $H^1(0, 0) \leq H^2(0, 0)$ <sup>10</sup>, then the firm starts sourcing from contract 2 immediately, i.e.  $\hat{Q}^1 = 0$ . If the marginal profit of sourcing the last feasible unit of input from contract 1 is higher than the marginal profit of sourcing the same from contract 2, i.e.  $H^1(\bar{Q}^1, 0) \geq H^2(\bar{Q}^1, 0)$ , then the firm switches to contract 2 only after exhausting the capacity of contract 1, i.e.  $\hat{Q}^1 = \bar{Q}^1$ . Otherwise, there exists a unique  $\hat{Q}^1 \in (0, \bar{Q}^1)$ . The exact amount of input sourced with this optimal contracting policy depends on the cost and revenue parameters. For example, the firm may prefer contract 1 over contract 2 for sourcing input up to  $\bar{Q}^1$ , i.e.  $H^1(\bar{Q}^1, 0) \geq H^2(\bar{Q}^1, 0)$  and  $\hat{Q}^1 = \bar{Q}^1$ , but it may not be optimal to source any input from contract 1 if the marginal profit of sourcing the first unit from contract 1 is not positive, i.e.  $H^1(0, 0) \leq 0$ .

It follows from Proposition 11 that, with stochastic demand, the optimal contracting decision follows a greedy policy only under special conditions:

**Corollary 1** *Let  $T_1^i \doteq -\beta^i - b^i + G(\max(b^i, h_1), h_3)$  and  $T_2^i \doteq H^i(\bar{Q}^1, 0)$  denote the two indices for contract  $i = 1, 2$  where  $H^i(x, y)$  is given in (16). The firm uses contract 2 (contract 1) only after contract 1's (contract 2's) capacity  $\bar{Q}^1$  ( $\bar{Q}^2$ ) is exhausted if  $T_1^1 \geq T_1^2$  and  $T_1^2 \geq T_2^2$  ( $T_1^1 < T_1^2$  and  $T_1^2 < T_2^2$ ).*

Corollary 1 characterizes two indices for each contract, one that is identical to the index of Proposition 10 in the deterministic demand case, and the other which depends on the first contract's maximum capacity. The optimal contracting decision follows a greedy policy only if the order of the contracts with respect to each of the two indices are consistent. If the firm uses a single-sourcing policy, i.e. if the firm needs to choose only one of the contracts, the preferred contract is characterized by Corollary 1 when the maximum capacity  $\bar{Q}^i$  of the preferred contract is not less than the other.

## 6 Conclusion

This paper provides insights on the optimal procurement decisions of a commodity processor for a single-input, multiple-output, fixed proportions technology setting with uncertainties in both input price and output demands. The central question analyzed is the structure of the optimal sourcing portfolio between spot markets and long-term contract markets, with the latter taken to be in the option contract form, with heterogeneous cost structure among available contracts. Our analysis contributes to the literature on contracting in the presence

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<sup>10</sup> $H^i(Q, 0)$  is an analogue of the first-order optimality condition given in (12) for contract  $i$ .

of spot markets (*i*) by characterizing the optimal contracting decisions of this commodity processor in the presence of multiple heterogeneous contracts under the fixed proportions technology model; (*ii*) by providing new comparative statics results and delineating the impact of the contract structure and the spot procurement access of the processor on these results.

In a single-contract setting, we analyze the impact of the input spot price and the output demand uncertainties on the optimal expected profit and the optimal contract volume of the commodity processor. Our results are summarized in Table 1, and demonstrate that the commodity processor benefits from a higher demand correlation, and a higher spot price variability but a lower demand variability. Our managerial insights, as summarized in §4.3, provide some rules of thumb for strategic supply management. Our most revealing finding is that some of these insights critically depend on the contract structure and the spot procurement access of the processor, and we explain why. A key implication of this finding is that using a forward or an option contract may lead to diametrically opposed operational choices in response to a change in uncertainties in the environment.

In a two-contract setting, we show that the optimal contracting decision follows a greedy policy under deterministic demands: The contracts are ranked in decreasing order of an index, which we characterize in closed form, and the second contract is used only after the capacity of the first contract is exhausted. With stochastic demands, we demonstrate that the optimal contracting decision is more complex, and follows a greedy policy only under special conditions. Under the greedy procurement policy, the order of the contracts is determined by two indices, which we also characterize in closed form. These results provide guidelines for the contract selection problem when the processor uses a single-sourcing policy.

There are a number of limitations to the present study that lead to open research questions. First, there is our simplifying assumption of a two-contract world with only two final products. Although the analysis will be more complex, we expect our main results to continue to hold in an  $m$ -contract world with  $n$ -products. Second, we assume that the contract parameters are exogenous. It would be interesting to analyze the impact of endogenous contract terms on the sourcing portfolio in an equilibrium setting, following the examples of Wu and Kleindorfer (2005), Dong and Liu (2007), Martínez-de-Albeniz and Simchi-Levi (2009), and Pei et al. (2010) who provide some results for the case of a single-

product firm. Third, we assume that the contracts are fully reliable, i.e. all the capacity reserved in the contract market can be used later on the spot day. In agricultural markets, the reserved capacities may correspond to the harvest, and may be prone to yield risks due to weather conditions and diseases. Moreover, the input yield may have an impact on the proportions of the final products. For example, if the input yield is sufficiently low, it may not be cost effective to process some of the final products (which translates into a zero proportion). This is relevant for markets that require large setup costs for processing, such as biofuels. It would be interesting to analyze the supplier yield issues, and their impact on the sourcing portfolio in the context of the fixed proportions technology.

Relaxing the assumptions we made on the production environment gives rise to a number of interesting possibilities, both in the theory of multi-product fixed proportions technology as well as in specific application areas in different commodity markets. For example, for tractability reasons, we assume that input spot price and output demand distributions are not correlated. With correlated spot price and demand distributions, we expect that the processor benefits from a lower correlation due to the optimal usage of the available procurement and production options. The impact of this correlation on the sourcing portfolio is expected to be more subtle, and needs further analysis. We assume that the final product prices are fixed and the processor faces a newsvendor problem in each product market. This is a reasonable assumption for many commodity markets with fixed proportions technology, but can be a limitation for some others. For example, in the palm industry, some of the final products (crude palm oil, palm kernel oil) are traded commodities, and the prices of these products are benchmarked to the spot prices. Moreover, the processor can buy or sell these products in the spot markets. Our model implicitly assumes that there is no product substitution among final products. This is a reasonable assumption for the majority of agricultural products, but is not applicable for some other commodity markets such as oil. As discussed in Dong et al. (2010), downward substitution among the final products, where a higher quality product can be converted into a lower quality product, plays a key role in the profitability of the processor. These comments and noted limitations suggest a number of open research questions.

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# TECHNICAL APPENDIX TO SUPPLY MANAGEMENT IN MULTI-PRODUCT FIRMS WITH FIXED PROPORTIONS TECHNOLOGY

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§A contains the proofs for our technical statements in the paper. A bivariate normal  $(D_1, D_2)$  can be expressed as the following transformation of two independent standard normal random variables  $Z_1$  and  $Z_2$ :  $D_1 = \sigma_1 \sqrt{\frac{1-\rho}{2}} Z_1 + \sigma_1 \sqrt{\frac{1+\rho}{2}} Z_2 + \mu_1$ ,  $D_2 = -\sigma_2 \sqrt{\frac{1-\rho}{2}} Z_1 + \sigma_2 \sqrt{\frac{1+\rho}{2}} Z_2 + \mu_2$ . We use  $\phi(\cdot)$  and  $\Phi(\cdot)$  denote the standard normal p.d.f and c.d.f., respectively. Throughout this appendix, we use the following relationships:  $\phi'(z) = -z\phi(z)$ ,  $\int_{-\infty}^v z\phi(z)dz = -\phi(v)$ ; and for a bivariate normal  $(D_1, D_2)$ ,  $P(D_1 \leq d_1, D_2 \leq d_2)$  is increasing in the demand correlation  $\rho$  for a fixed  $(d_1, d_2)$  as follows from the Slepian's inequality (Tong 1990, p. 21).

## A Main Proofs

**Proof of Proposition 1 :** Let  $\Gamma_j$  denote the objective function in (3). We obtain  $\frac{\partial \Gamma_j}{\partial x_j} = (p_j + r_j) - (c_j + \bar{s}_j)$  for  $x_j \leq D_j$  and  $\frac{\partial \Gamma_j}{\partial x_j} = \underline{s}_j - (c_j + \bar{s}_j)$  for  $x_j > D_j$ . Since  $(p_j + r_j) \geq \underline{s}_j$  by assumption,  $\Gamma_j$  is a piecewise linear function and concave in  $x_j$ . It is easy to establish  $x_j^*$  as given in (1). Substituting  $x_j^*$  in  $\Gamma_j$ , we obtain

$$\Psi_j(\mathbf{z}^{[j]}) = \begin{cases} -r_j D_j + \bar{s}_j a_j \mathbf{1}' \mathbf{z}^{[j]} & \text{if } c_j + \bar{s}_j \geq p_j + r_j \geq \underline{s}_j \\ (p_j - c_j) \min(a_j \mathbf{1}' \mathbf{z}^{[j]}, D_j) - r_j (D_j - a_j \mathbf{1}' \mathbf{z}^{[j]})^+ + \bar{s}_j (a_j \mathbf{1}' \mathbf{z}^{[j]} - D_j)^+ & \text{if } p_j + r_j \geq c_j + \bar{s}_j \geq \underline{s}_j \\ p_j \min(a_j \mathbf{1}' \mathbf{z}^{[j]}, D_j) - r_j (D_j - a_j \mathbf{1}' \mathbf{z}^{[j]})^+ + \underline{s}_j (a_j \mathbf{1}' \mathbf{z}^{[j]} - D_j)^+ - c_j a_j \mathbf{1}' \mathbf{z}^{[j]} & \text{if } p_j + r_j \geq \underline{s}_j \geq c_j + \bar{s}_j. \end{cases}$$

Using the identities  $a_j \mathbf{1}' \mathbf{z}^{[j]} = D_j - (D_j - a_j \mathbf{1}' \mathbf{z}^{[j]})^+ + (a_j \mathbf{1}' \mathbf{z}^{[j]} - D_j)^+$ ,  $D_j = a_j \mathbf{1}' \mathbf{z}^{[j]} - (a_j \mathbf{1}' \mathbf{z}^{[j]} - D_j)^+ - D_j)^+ + (D_j - a_j \mathbf{1}' \mathbf{z}^{[j]})^+$ ,  $\min(D_j, a_j \mathbf{1}' \mathbf{z}^{[j]}) = D_j - (D_j - a_j \mathbf{1}' \mathbf{z}^{[j]})^+ = a_j \mathbf{1}' \mathbf{z}^{[j]} - (a_j \mathbf{1}' \mathbf{z}^{[j]} - D_j)^+$  and defining  $\hat{s}_j \doteq \max(c_j + \bar{s}_j, \underline{s}_j)$ ,  $\hat{p}_j \doteq \max(p_j, c_j + \bar{s}_j - r_j)$ ,  $\Psi_j$  above can be rewritten as (5). ■

**Proof of Proposition 2 :** With respect to the ordering among  $\frac{D_1}{a_1} \leq \frac{D_2}{a_2}$  and  $\sum_{k=[1]}^{[i]} q^k \quad \forall [i]$ , we have 10 cases to analyze to explicitly characterize  $z^*$ . For brevity, we only analyze the case  $q^{[1]} \leq \frac{D_1}{a_1} \leq q^{[1]} + q^{[2]} \leq \frac{D_2}{a_2} \leq q^{[1]} + q^{[2]} + q^{[3]}$ . The other cases can be analyzed in a similar fashion. Substituting  $\Psi_j(z)$  from (6), and using the identities  $\min(x, y) = x - (x - y)^+$  and  $x = y - (y - x)^+ + (x - y)^+$ , we obtain  $\Pi^{[i]}(z) = (\hat{\mathbf{p}} - \mathbf{c} - b^{[i]} \mathbf{1} - \omega \mathbf{1})' \mathbf{D} + \sum_{k=[1]}^{[i]} (b^{[i]} - b^k) q^k -$

$[(\hat{\mathbf{p}} + \mathbf{r} - \mathbf{c} - b^{[i]} \mathbf{1} - \omega \mathbf{1})'(\mathbf{D} - \mathbf{a}z)^+ + (\mathbf{c} + b^{[i]} \mathbf{1} + \omega \mathbf{1} - \hat{\mathbf{s}})'(\mathbf{a}z - \mathbf{D})^+]$ . It follows from (6) that

$$\frac{\partial \Pi(z)}{\partial z} = \begin{cases} h_3 - b^{[1]} & \text{if } 0 \leq z \leq q^{[1]} \\ h_3 - b^{[2]} & \text{if } q^{[1]} < z \leq \frac{D_1}{a_1} \\ h_2 - b^{[2]} & \text{if } \frac{D_1}{a_1} < z \leq q^{[1]} + q^{[2]} \\ h_2 - b^{[3]} & \text{if } q^{[1]} + q^{[2]} < z \leq \frac{D_2}{a_2} \\ h_1 - b^{[3]} & \text{if } \frac{D_2}{a_2} < z \leq q^{[1]} + q^{[2]} + q^{[3]}. \end{cases}$$

It is easy to establish that  $\lim_{z \rightarrow \bar{z}^-} \frac{\partial \Pi(z)}{\partial z} > \lim_{z \rightarrow \bar{z}^+} \frac{\partial \Pi(z)}{\partial z}$  for  $\bar{z} \in \{q^{[1]}, \frac{D_1}{a_1}, q^{[1]} + q^{[2]}, \frac{D_2}{a_2}\}$ , thus  $\Pi(z)$  is a piecewise linear and concave in  $z$ . The optimal processing level  $z^*$  is characterized by

$$z^* = \begin{cases} 0 & \text{if } h_3 \leq b^{[1]} \\ q^{[1]} & \text{if } b^{[1]} < h_3 \leq b^{[2]} \\ \frac{D_1}{a_1} & \text{if } h_2 < b^{[2]} \leq h_3 \\ q^{[1]} + q^{[2]} & \text{if } b^{[2]} < h_2 \leq b^{[3]} \\ \frac{D_2}{a_2} & \text{if } h_1 < b^{[3]} \leq h_2 \\ q^{[1]} + q^{[2]} + q^{[3]} & \text{if } b^{[3]} < h_1 \end{cases} \quad (17)$$

which is identical to (7) for the case  $q^{[1]} \leq \frac{D_1}{a_1} \leq q^{[1]} + q^{[2]} \leq \frac{D_2}{a_2} \leq q^{[1]} + q^{[2]} + q^{[3]}$ . ■

**Proof of Proposition 3 :** The proof is omitted. ■

**Proof of Proposition 4 :** Substituting  $K = 0$  in Proposition 3, the optimality condition is given by  $Foc(Q) = -\beta - b + P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} \leq Q\right) \max(b, h_1) + P\left(\frac{D_1}{a_1} \leq Q, \frac{D_2}{a_2} > Q\right) \max(b, h_2^1) + P\left(\frac{D_1}{a_1} > Q, \frac{D_2}{a_2} \leq Q\right) \max(b, h_2^2) + P\left(\frac{D_1}{a_1} > Q, \frac{D_2}{a_2} > Q\right) h_3$ . From the implicit function theorem, we have  $sgn\left(\frac{\partial Q^*}{\partial \tau}\right) = sgn\left(\frac{\partial Foc(Q^*)}{\partial \tau}\right)$  for a given parameter  $\tau$ . For notational convenience, we define  $A \doteq \{D_1 \leq a_1 Q^*\}$  and  $B \doteq \{D_2 \leq a_2 Q^*\}$ . Transforming  $(D_1, D_2)$  as a function of standard normal variables  $(Z_1, Z_2)$ , we obtain

$$P(AB) = \int_{-\infty}^{\bar{Z}_1} \Phi(Y) \phi(Z_1) dZ_1 + \int_{\bar{Z}_1}^{+\infty} \Phi(X) \phi(Z_1) dZ_1$$

where  $X = (a_1 Q^* - \mu_1 - \sigma_1 \sqrt{\frac{1-\rho}{2}} Z_1) / \sigma_1 \sqrt{\frac{1+\rho}{2}}$  and  $Y = (a_2 Q^* - \mu_2 + \sigma_2 \sqrt{\frac{1-\rho}{2}} Z_1) / \sigma_2 \sqrt{\frac{1+\rho}{2}}$ , and  $\bar{Z}_1$  is the unique solution to  $X(Z_1) = Y(Z_1)$ .  $P(\bar{A}\bar{B}), P(\bar{A}B), P(\bar{A}\bar{B})$  can be established in a similar fashion.

For the impact of  $\mu_1$  on  $Q^*$ , we obtain  $\frac{\partial Foc(Q^*)}{\partial \mu_1} =$

$$[\max(b, h_1) - \max(b, h_2^2)] \int_{\bar{Z}_1}^{+\infty} \phi(X) \frac{-1}{\sigma_1 \sqrt{\frac{1+\rho}{2}}} \phi(Z_1) dZ_1 + [\max(b, h_2^1) - h_3] \int_{-\infty}^{\bar{Z}_1} \phi(X) \frac{-1}{\sigma_1 \sqrt{\frac{1+\rho}{2}}} \phi(Z_1) dZ_1 \geq 0.$$

Therefore,  $Q^*$  is increasing in  $\mu_1$ . The impact of  $\mu_2$  on  $Q^*$  can be established in a similar fashion.

For the impact of  $\sigma_1$  on  $Q^*$ , we obtain

$$\frac{\partial Foc(Q^*)}{\partial \sigma_1} = [\max(b, h_1) - \max(b, h_2^2)] \int_{\bar{Z}_1}^{+\infty} \phi(X) \frac{\partial X}{\partial \sigma_1} \phi(Z_1) dZ_1 + [\max(b, h_2^1) - h_3] \int_{-\infty}^{\bar{Z}_1} \phi(X) \frac{\partial X}{\partial \sigma_1} \phi(Z_1) dZ_1$$

where  $\frac{\partial X}{\partial \sigma_1} = \frac{-1}{\sigma_1^2} \left( \frac{a_1 Q^* - \mu_1}{\sqrt{1 + \frac{\rho}{2}}} \right)$ . It is easy to verify that  $\frac{\partial \text{Foc}(Q^*)}{\partial \sigma_1} \geq 0$ , when  $Q^* \geq \frac{\mu_1}{a_1}$  and  $Q^*$  increases in  $\sigma_1$  (and decreases otherwise). The impact of  $\sigma_2$  on  $Q^*$  can be established in a similar fashion.

To analyze the impact of  $\rho$ , we will use the optimality condition given in (10). Using the identities,  $P(\bar{A}) = 1 - P(A)$ ,  $P(\bar{B}) = 1 - P(B)$ ,  $P(A\bar{B}) = P(A) - P(AB)$ ,  $P(\bar{A}B) = P(B) - P(AB)$ ,  $P(\bar{A}\bar{B}) = 1 - P(A) - P(B) + P(AB)$ , we obtain

$$\frac{\partial \text{Foc}(Q^*)}{\partial \rho} = [(b - h_1)^+ - (b - h_2^1)^+ - (b - h_2^2)^+] \frac{\partial P(AB)}{\partial \rho} \geq 0$$

as follows from the Slepian inequality, where the equality only holds for  $b \leq h_1$ . ■

**Proof of Proposition 5 :** To show the impact on  $V^*(Q)$ , we will prove that the results hold for  $V(Q)$  for any  $Q$ . We have  $V(Q) = \mathbb{E}_{\tilde{\mathbf{D}}}[\pi(Q; \tilde{\mathbf{D}})] - \beta Q$ . The form of  $\pi(Q; \tilde{\mathbf{D}})$  depends on the ordering between  $b$  and  $h_1 \leq h_2^1 \leq h_2^2 \leq h_3$ , thus we have four cases to consider. We will only analyze the case  $h_1 \leq h_2^1 \leq b \leq h_2^2 \leq h_3$ . The other cases can be analyzed in a similar fashion. We obtain

$$\pi(Q; \mathbf{D}) = \begin{cases} (h_2^2 - b - a_1 r_1) \frac{D_1}{a_1} + (\hat{p}_2 - \hat{s}_2) D_2 & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_2}{a_2} \leq \frac{D_1}{a_1} \leq Q \right\} \\ (h_3 - b - a_1 r_1) \frac{D_1}{a_1} - r_2 D_2 & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_1}{a_1} < \frac{D_2}{a_2} \leq Q \right\} \\ (h_3 - b - a_1 r_1) \frac{D_1}{a_1} - r_2 D_2 & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_1}{a_1} \leq Q < \frac{D_2}{a_2} \right\} \\ -r_1 D_1 + (\hat{p}_2 - \hat{s}_2) D_2 + (h_2^2 - b) Q & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_2}{a_2} \leq Q < \frac{D_1}{a_1} \right\} \\ -r_1 D_1 - r_2 D_2 + (h_3 - b) Q & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : Q < \frac{D_2}{a_2} \leq \frac{D_1}{a_1} \right\} \\ -r_1 D_1 - r_2 D_2 + (h_3 - b) Q & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : Q < \frac{D_1}{a_1} < \frac{D_2}{a_2} \right\} \end{cases} \quad (18)$$

For the impact of  $\mu_j$  on  $V(Q)$ , we use the following result Müller (2001):

**Lemma A.1** *Let  $\tilde{\mathbf{D}}$  ( $\tilde{\mathbf{D}}$ ) to have a bivariate normal distribution with mean  $\underline{\boldsymbol{\mu}}$  ( $\underline{\boldsymbol{\mu}}$ ) and a covariance matrix  $\Sigma$  ( $\Sigma$ ). If  $\underline{\boldsymbol{\mu}} \leq \underline{\boldsymbol{\mu}}$  and  $\Sigma = \Sigma$ , then  $\underline{\mathbf{D}} \geq \mathbf{D}$  in increasing order, so that  $\mathbb{E}[f(\underline{\mathbf{D}})] \leq \mathbb{E}[f(\mathbf{D})]$  for any increasing function  $f$ .*

Using (18), it is easy to show that  $\frac{\partial \pi(Q; \mathbf{D})}{\partial D_j} \geq 0$  for  $r_j = 0$ , thus the result follows from Lemma A.1.

For the impact of  $\sigma$  for  $\sigma = \sigma_1 = \sigma_2$  on  $V(Q)$ , we use the following result Müller (2001):

**Lemma A.2** *Let  $\tilde{\mathbf{D}}$  ( $\tilde{\mathbf{D}}$ ) to have a bivariate normal distribution with mean  $\underline{\boldsymbol{\mu}}$  ( $\underline{\boldsymbol{\mu}}$ ) and a covariance matrix  $\Sigma$  ( $\Sigma$ ) with  $\sigma = \sigma_1 = \sigma_2$  ( $\underline{\sigma} = \underline{\sigma}_1 = \underline{\sigma}_2$ ). If  $\underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\mu}}$  and  $\underline{\sigma} \leq \underline{\sigma}$ , then  $\underline{\mathbf{D}} \leq \mathbf{D}$  in the concave order, so that  $\mathbb{E}[f(\underline{\mathbf{D}})] \leq \mathbb{E}[f(\mathbf{D})]$  for any concave function  $f$ .*

As follows from Lemma A.2, to show that  $V(Q)$  is decreasing in demand variability, it is sufficient to show that  $\pi(Q; \mathbf{D})$  is jointly concave in  $(D_1, D_2)$ . We have  $\pi(Q; \mathbf{D}) = \max_{0 \leq z \leq Q} \Pi(z)$  where  $\Pi(z)$  is given by

$$\Pi(z) = \begin{cases} (\hat{p}_1 - \hat{s}_1) D_1 + (\hat{p}_2 - \hat{s}_2) D_2 + (h_1 - b) z & \text{if } \mathbf{D} \in \Omega_1 \doteq \{ \mathbf{D} : D_1 \leq a_1 z, D_2 \leq a_2 z \} \\ (\hat{p}_1 - \hat{s}_1) D_1 - r_2 D_2 + (h_2^1 - b) z & \text{if } \mathbf{D} \in \Omega_2 \doteq \{ \mathbf{D} : D_1 \leq a_1 z, D_2 > a_2 z \} \\ -r_1 D_1 + (\hat{p}_2 - \hat{s}_2) D_2 + (h_2^2 - b) z & \text{if } \mathbf{D} \in \Omega_3 \doteq \{ \mathbf{D} : D_1 > a_1 z, D_2 \leq a_2 z \} \\ -r_1 D_1 - r_2 D_2 + (h_3 - b) z & \text{if } \mathbf{D} \in \Omega_4 \doteq \{ \mathbf{D} : D_1 > a_1 z, D_2 > a_2 z \} \end{cases}$$

As follows from the basic optimization theory, concavity is preserved under maximization, thus it is sufficient to show that  $\Pi(z)$  is jointly concave in  $(D_1, D_2)$ . This can be easily established by using the definition of concave function:  $\Pi$  is jointly concave in  $(D_1, D_2)$ , if  $\Pi\left(\lambda\mathbf{D}^{0'} + (1-\lambda)\mathbf{D}^{1'}\right) \geq \lambda\Pi(\mathbf{D}^{0'}) + (1-\lambda)\Pi(\mathbf{D}^{1'})$  for  $\lambda \in [0, 1]$  where  $\mathbf{D}^{0'} = (D_1^0, D_2^0) \in \Omega_i$  and  $\mathbf{D}^{1'} = (D_1^1, D_2^1) \in \Omega_j$  for  $i, j \in \{1, 2, 3, 4\}$ .

For the impact of  $\rho$  on  $V(Q)$ , we use the following result Müller (2001):

**Lemma A.3** *Let  $\tilde{\mathbf{D}}$  ( $\underline{\mathbf{D}}$ ) to have a bivariate normal distribution with mean  $\boldsymbol{\mu}$  ( $\underline{\boldsymbol{\mu}}$ ) and covariance matrix  $\Sigma$  ( $\underline{\Sigma}$ ). If  $\boldsymbol{\mu} = \underline{\boldsymbol{\mu}}$ ,  $\tilde{\mathbf{D}}$  and  $\underline{\mathbf{D}}$  have the same marginal distributions,  $\Sigma_{ij} \leq \underline{\Sigma}_{ij}$ , then  $\tilde{\mathbf{D}} \leq \underline{\mathbf{D}}$  in the supermodular order, i.e.  $\mathbb{E}[f(\tilde{\mathbf{D}})] \leq \mathbb{E}[f(\underline{\mathbf{D}})]$  for any supermodular function  $f$ .*

It follows from Lemma A.3 that increasing  $\rho$  leads to another bivariate normal distribution that is preferred over  $\tilde{\mathbf{D}}$  in the supermodular order. It is sufficient to show that  $\pi(Q; \mathbf{D})$  is supermodular in  $\tilde{\mathbf{D}}$ . To prove supermodularity, it is sufficient to show  $\frac{\partial^2 \pi(Q; \mathbf{D})}{\partial D_1 \partial D_2} \geq 0$ . Using (18), we obtain

$$\frac{\partial \pi(Q; \mathbf{D})}{\partial D_1} = \begin{cases} (h_2^2 - b - a_1 r_1) \frac{1}{a_1} & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_2}{a_2} \leq \frac{D_1}{a_1} \leq Q \right\} \\ (h_3 - b - a_1 r_1) \frac{1}{a_1} & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_1}{a_1} < \frac{D_2}{a_2} \leq Q \right\} \\ (h_3 - b - a_1 r_1) \frac{1}{a_1} & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_1}{a_1} \leq Q < \frac{D_2}{a_2} \right\} \\ -r_1 & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : \frac{D_2}{a_2} \leq Q < \frac{D_1}{a_1} \right\} \\ -r_1 & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : Q < \frac{D_2}{a_2} \leq \frac{D_1}{a_1} \right\} \\ -r_1 & \text{if } \mathbf{D} \in \left\{ \mathbf{D} : Q < \frac{D_1}{a_1} < \frac{D_2}{a_2} \right\} \end{cases} \quad (19)$$

It is easy to show that for a given  $D_1$ ,  $\frac{\partial \pi(Q; \mathbf{D})}{\partial D_1}$  increases as  $D_2$  increases. For example, for a given  $D_1$  such that  $D_1 < a_1 Q$ , as  $D_2$  increases  $\frac{\partial \pi(Q; \mathbf{D})}{\partial D_1}$  traces the first three regions in (19) in this order. Since  $h_3 > h_2^2$ ,  $\frac{\partial \pi(Q; \mathbf{D})}{\partial D_1 \partial D_2} \geq 0$ . This concludes the proof. ■

**Proof of Proposition 6** : The proof is omitted. ■

**Proof of Proposition 7** : From the implicit function theorem, we have  $\text{sgn}\left(\frac{\partial Q^*}{\partial \tau}\right) = \text{sgn}\left(\frac{\partial \text{Foc}(Q^*)}{\partial \tau}\right)$  for a given parameter  $\tau$  where the optimality condition  $\text{Foc}(Q)$  is as given in (12). For notational convenience, we use  $A = \{D_1 \leq a_1 Q^*\}$  and  $B = \{D_2 \leq a_2 Q^*\}$ . We use the following Lemma throughout the proof (and the proof of the Lemma is omitted for brevity):

**Lemma A.4** *Define  $G(\lambda_0, \lambda_1) \doteq \mathbb{E}[\min(\max(\tilde{P}^S, \lambda_0), \lambda_1)]$  for  $\lambda_0 \leq \lambda_1$ .  $G(\lambda_0, \lambda_1)$  increases in  $\lambda_0$  and  $\lambda_1$ . If  $\tilde{P}^S$  follows a normal distribution with mean  $\mu_S$  and standard deviation  $\sigma_S$ , we have  $G(\lambda_0, \lambda_1) = \sigma_S \left( L\left(\frac{\lambda_0 - \mu_S}{\sigma_S}\right) - L\left(\frac{\lambda_1 - \mu_S}{\sigma_S}\right) \right)$  where  $L(\eta) = \int_{-\infty}^{\eta} (\eta - z)\phi(z)dz$  is the standard-normal loss function.  $G(\lambda_0, \lambda_1)$  increases in  $\mu_S$ ; and increases in  $\sigma_S$  if  $\mu_S \leq \frac{\lambda_0 + \lambda_1}{2}$  (and decreases otherwise).*

Since  $G(\lambda_0, \lambda_1)$  is increasing in  $\mu_S$  as follows from Lemma A.4,  $Q^*$  increases in  $\mu_S$ .

For the impact of  $\sigma_S$  on  $Q^*$ , we only provide the proof for the case  $h_2^1 \leq b \leq h_2^2 \leq h_3$ . Using  $G(\lambda_0, \lambda_1)$  expression for the standard normal distribution (as given in Lemma A.4), we obtain

$$\frac{\partial \text{Foc}(Q^*)}{\partial \sigma_S} = P(\bar{A}B) \left[ \phi\left(\frac{b-\mu_S}{\sigma_S}\right) - \phi\left(\frac{h_2^2-\mu_S}{\sigma_S}\right) \right] + P(\bar{A}\bar{B}) \left[ \phi\left(\frac{b-\mu_S}{\sigma_S}\right) - \phi\left(\frac{h_3-\mu_S}{\sigma_S}\right) \right] \quad (20)$$

For  $\mu_S \leq \frac{b+h_2^2}{2}$ , we have  $\frac{\partial \text{Foc}(Q^*)}{\partial \sigma_S} > 0$ , for  $\mu_S > \frac{b+h_3}{2}$  and  $\frac{\partial \text{Foc}(Q^*)}{\partial \sigma_S} < 0$ . Next, we analyze the case  $\mu_S \in [\frac{b+h_2^2}{2}, \frac{b+h_3}{2}]$ . We define  $H(\mu_S) \doteq \frac{\partial \text{Foc}(Q^*)}{\partial \sigma_S}$ . Taking the derivative of  $H(\mu_S)$  w.r.t to  $\mu_S$ , after some algebra, we obtain

$$\frac{\partial H(\mu_S)}{\partial \mu_S} = \left(\frac{b-\mu_S}{\sigma_S}\right) \frac{1}{\sigma_S} \left[ H(\mu_S) + \frac{(h_2^2-b)}{(\mu_S-b)} \phi\left(\frac{h_2^2-\mu_S}{\sigma_S}\right) + \frac{(h_3-b)}{(\mu_S-b)} \phi\left(\frac{h_3-\mu_S}{\sigma_S}\right) \right].$$

Since  $h_2^1 \leq b \leq h_2^2$  and  $\mu_S \in [\frac{b+h_2^2}{2}, \frac{b+h_3}{2}]$  by assumption, we have  $b < \mu_S$ . Therefore,  $\frac{\partial H(\mu_S)}{\partial \mu_S} < 0$  for  $H(\mu_S) \geq 0$ , and the sign of  $\frac{\partial H(\mu_S)}{\partial \mu_S}$  is indeterminate for  $H(\mu_S) < 0$ . Since  $H(\mu_S)$  is continuous in  $\mu_S$ , and we have  $H(\mu_S) > 0$  for  $\mu_S \leq \frac{b+h_2^2}{2}$ ;  $H(\mu_S) < 0$  for  $\mu_S > \frac{b+h_3}{2}$ , thus there exist at least one  $\hat{\mu}_S$  such that  $H(\hat{\mu}_S) = 0$  and  $H(\mu_S) > 0$  for  $\mu_S < \hat{\mu}_S$ . Since  $\frac{\partial H(\mu_S)}{\partial \mu_S} < 0$  for  $H(\mu_S) = 0$ ,  $\hat{\mu}_S$  is unique. In summary, we have  $\frac{\partial Q^*}{\partial \sigma_S} > 0$  for  $\mu_S < \hat{\mu}_S$  and  $\frac{\partial Q^*}{\partial \sigma_S} \leq 0$  for  $\mu_S \geq \hat{\mu}_S$ . We now analyze the impact of  $b$  on  $\hat{\mu}_S(b)$ . From (20), for a given  $\hat{\mu}_S$ , we obtain

$$\frac{\partial H(\hat{\mu}_S)}{\partial b} = [P(\bar{A}B) + P(\bar{A}\bar{B})] \frac{1}{\sigma_S} \left[ \frac{-(b-\hat{\mu}_S)}{\sigma_S} \right] \phi\left(\frac{b-\hat{\mu}_S}{\sigma_S}\right) > 0,$$

as  $b < \hat{\mu}_S$ . Thus,  $\hat{\mu}_S(b)$  increases in  $b$ . ■

**Proof of Proposition 8 :** The proofs for the impact of  $\mu_j$  and  $\sigma_j$  on  $Q^*$  is similar to the proof of Proposition 4, thus, are omitted. For the impact of  $\rho$  on  $Q^*$ , relying on the notation  $A = \{D_1 \leq a_1 Q^*\}$  and  $B = \{D_2 \leq a_2 Q^*\}$ , and using the identities  $P(\bar{A}) = 1 - P(A)$ ,  $P(\bar{B}) = 1 - P(B)$ ,  $P(\bar{A}\bar{B}) = P(A) - P(AB)$ ,  $P(\bar{A}B) = P(B) - P(AB)$ ,  $P(\bar{A}\bar{B}) = 1 - P(A) - P(B) + P(AB)$  on (12), we obtain

$$\frac{\partial \text{Foc}(Q^*)}{\partial \rho} = \left[ \max(b, h_1) - G(\max(b, h_1), \max(b, h_2^1)) - G(\max(b, h_1), \max(b, h_2^2)) + G(\max(b, h_1), h_3) \right] \frac{\partial P(AB)}{\partial \rho}$$

Since  $\frac{\partial P(AB)}{\partial \rho}$  is increasing in  $\rho$  as follows from the Slepian's inequality, the sign of the impact of  $\rho$  is given by the sign of the multiplier in parenthesis which we define as  $M(b)$ , i.e.

$$M(b) \doteq \begin{cases} h_1 + G(h_1, h_3) - G(h_1, h_2^1) - G(h_1, h_2^2) & \text{if } b \in [0, h_1) \\ b + G(b, h_3) - G(b, h_2^1) - G(b, h_2^2) & \text{if } b \in [h_1, h_2^1) \\ b + G(b, h_3) - b - G(b, h_2^2) & \text{if } b \in [h_2^1, h_2^2) \\ b + G(b, h_3) - b - b & \text{if } b \in [h_2^2, h_3). \end{cases}$$

It is easy to show that  $M(b) \leq 0$  for  $b \leq h_1$  and  $M(b) \geq 0$  for  $b \geq h_2^1$ . For  $b \in [h_1, h_2^1)$ , we have  $M(b) = b + G(b, h_3) - G(b, h_2^1) - G(b, h_2^2)$ . Using  $G(\lambda_0, \lambda_1)$  expression for the standard normal distribution (as given in Lemma A.4), we obtain  $\frac{\partial M}{\partial b} = 1 - \Phi\left(\frac{b-\mu_S}{\sigma_S}\right) > 0$ . Thus, there exists a unique  $\bar{b} \in [h_1, h_2^1)$  such that  $M(b) \leq 0$ , when  $b \leq \bar{b}$  and  $M(b) \geq 0$ , when  $b \geq \bar{b}$ . This concludes the proof. ■

**Proof of Proposition 9 :** To show the impact on  $V^*(Q)$ , we will prove that the results hold for  $V(Q)$  for any  $Q$ . We have  $V(Q) = \mathbb{E}[\pi(Q; \tilde{\mathbf{D}}, \tilde{P}^S)] - \beta Q$ . To proofs for the impact of  $\mu_j$ ,  $\sigma_j$  and  $\rho$  can be established in a similar fashion with Proposition 5.<sup>11</sup> Therefore, we only prove the impact of  $\mu_S$  and  $\sigma_S$  on  $V(Q)$ . Let  $H(\tilde{D}_1, \tilde{D}_2)$  and  $F(\tilde{P}^S)$  denote the c.d.f of  $\tilde{\mathbf{D}}$  and  $\tilde{P}^S$ , respectively. We have  $\mathbb{E}[\pi(Q)] = \mathbb{E}_{\tilde{P}^S} [\mathbb{E}_{\tilde{\mathbf{D}}} [\pi(Q)]] = \int \left[ \iint \pi(Q, \tilde{D}_1, \tilde{D}_2; \tilde{P}^S) dH(\tilde{D}_1, \tilde{D}_2) \right] dF(\tilde{P}^S)$ . We use the following result from Müller (2001):

**Lemma A.5** *Let  $\tilde{P}^S$  ( $\underline{\tilde{P}}^S$ ) to have a normal distribution with mean  $\mu_S$  ( $\underline{\mu}_S$ ) and standard deviation  $\sigma_S$  ( $\underline{\sigma}_S$ ). If  $\mu_S \leq \underline{\mu}_S$  and  $\sigma_S = \underline{\sigma}_S$ , then,  $\tilde{P}^S \geq \underline{\tilde{P}}^S$  in the decreasing order, so that  $\mathbb{E}[f(\tilde{P}^S)] \geq \mathbb{E}[f(\underline{\tilde{P}}^S)]$  for any decreasing function  $f$ . If  $\mu_S = \underline{\mu}_S$  and  $\sigma_S \leq \underline{\sigma}_S$ , then,  $\tilde{P}^S \leq \underline{\tilde{P}}^S$  in the convex order, so that  $\mathbb{E}[g(\tilde{P}^S)] \leq \mathbb{E}[g(\underline{\tilde{P}}^S)]$  for any convex function  $g$ .*

As follows from Lemma A.5, it is sufficient to show that  $\mathbb{E}_{\tilde{\mathbf{D}}} [\pi(Q)]$  is convex decreasing in  $P^S$ . We will show that  $\pi(Q)$  is convex decreasing in  $P^S$  for a given  $\mathbf{D}$ . The form of  $\pi(Q)$  depends on the ordering between  $b$  and  $h_1 \leq h_2^1 \leq h_2^2 \leq h_3$ , we have four cases to consider. We will only analyze the case  $h_1 \leq h_2^1 \leq b \leq h_2^2 \leq h_3$ , paralleling our analysis in Proposition 5. The other cases can be analyzed in a similar fashion. The optimal stage 1 profit  $\pi(Q; \mathbf{D}, P^S)$  is given by

$$= \begin{cases} \Pi^1(K)\chi_1 + \Pi^1(K)\chi_2 + \Pi^1(K)\chi_3 + \Pi^1(K)\chi_4 + \Pi^1(K)\chi_5 + \Pi^1(K)\chi_6 & \text{if } P^S \in [0, h_1) \\ \Pi^1\left(\frac{D_1}{a_1}\right)\chi_1 + \Pi^1\left(\frac{D_2}{a_2}\right)\chi_2 + \Pi^1\left(\frac{D_2}{a_2}\right)\chi_3 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_4 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_5 + \Pi^1\left(\frac{D_2}{a_2}\right)\chi_6 & \text{if } P^S \in [h_1, h_2^1) \\ \Pi^1\left(\frac{D_1}{a_1}\right)\chi_1 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_2 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_3 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_4 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_5 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_6 & \text{if } P^S \in [h_2^1, b) \\ \Pi^1\left(\frac{D_1}{a_1}\right)\chi_1 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_2 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_3 + \Pi^2\left(\frac{D_1}{a_1}\right)\chi_4 + \Pi^2\left(\frac{D_1}{a_1}\right)\chi_5 + \Pi^2\left(\frac{D_1}{a_1}\right)\chi_6 & \text{if } P^S \in [b, h_2^2) \\ \Pi^1\left(\frac{D_1}{a_1}\right)\chi_1 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_2 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_3 + \Pi^1(Q)\chi_4 + \Pi^2\left(\frac{D_2}{a_2}\right)\chi_5 + \Pi^2\left(\frac{D_1}{a_1}\right)\chi_6 & \text{if } P^S \in [h_2^2, h_3) \\ \Pi^1\left(\frac{D_1}{a_1}\right)\chi_1 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_2 + \Pi^1\left(\frac{D_1}{a_1}\right)\chi_3 + \Pi^1(Q)\chi_4 + \Pi^1(Q)\chi_5 + \Pi^1(Q)\chi_6 & \text{if } P^S \in [h_3, +\infty) \end{cases}$$

where spot procurement capacity  $K$  is a very large number (as we are analyzing the unlimited spot procurement case),  $\Pi^i(z)$  is as given in (6), and the indicator  $\chi_i$  is characterized by

$$\begin{aligned} \chi_1 &\doteq \chi\left(\mathbf{D} : \frac{D_2}{a_2} \leq \frac{D_1}{a_1} \leq Q\right), & \chi_2 &\doteq \chi\left(\mathbf{D} : \frac{D_1}{a_1} < \frac{D_2}{a_2} \leq Q\right), & \chi_3 &\doteq \chi\left(\mathbf{D} : \frac{D_1}{a_1} \leq Q < \frac{D_2}{a_2}\right) \\ \chi_4 &\doteq \chi\left(\mathbf{D} : \frac{D_2}{a_2} \leq Q < \frac{D_1}{a_1}\right), & \chi_5 &\doteq \chi\left(\mathbf{D} : Q < \frac{D_2}{a_2} \leq \frac{D_1}{a_1}\right), & \chi_6 &\doteq \chi\left(\mathbf{D} : Q < \frac{D_1}{a_1} < \frac{D_2}{a_2}\right) \end{aligned}$$

We obtain

$$\frac{\partial \pi(Q; \mathbf{D}, P^S)}{\partial P^S} = \begin{cases} -K\chi_1 - K\chi_2 - K\chi_3 - K\chi_4 - K\chi_5 - K\chi_6 & \text{if } P^S \in [0, h_1) \\ -\frac{D_1}{a_1}\chi_1 - \frac{D_2}{a_2}\chi_2 - \frac{D_2}{a_2}\chi_3 - \frac{D_1}{a_1}\chi_4 - \frac{D_1}{a_1}\chi_5 - \frac{D_2}{a_2}\chi_6 & \text{if } P^S \in [h_1, h_2^1) \\ -\frac{D_1}{a_1}\chi_1 - \frac{D_1}{a_1}\chi_2 - \frac{D_1}{a_1}\chi_3 - \frac{D_1}{a_1}\chi_4 - \frac{D_1}{a_1}\chi_5 - \frac{D_1}{a_1}\chi_6 & \text{if } P^S \in [h_2^1, b) \\ -\left(\frac{D_1}{a_1} - Q\right)\chi_4 - \left(\frac{D_1}{a_1} - Q\right)\chi_5 - \left(\frac{D_1}{a_1} - Q\right)\chi_6 & \text{if } P^S \in [b, h_2^2) \\ -\left(\frac{D_2}{a_2} - Q\right)\chi_5 - \left(\frac{D_1}{a_1} - Q\right)\chi_6 & \text{if } P^S \in [h_2^2, h_3) \\ 0 & \text{if } P^S \in [h_3, +\infty) \end{cases}$$

<sup>11</sup>The analysis is more involved than the one in Proposition 5 as the stage 1 profit also depends on the spot prize realization, but same stochastic order results can be used to obtain the results.

It directly follows that  $\frac{\partial \pi(Q; \mathbf{D}, P^S)}{\partial P^S} \leq 0$ . We will only establish the convexity proof for one of the demand scenarios, the rest can be proven in a similar fashion. Assume that  $\mathbf{D}$  is such that  $Q < \frac{D_2}{a_2} \leq \frac{D_1}{a_1}$ , i.e. we have  $\chi_5 = 1$ , and the other indicators are 0. In this case,  $\frac{\partial \pi(Q; \mathbf{D}, P^S)}{\partial P^S}$  yields,

$$\frac{\partial \pi(Q; \mathbf{D}, P^S)}{\partial P^S} = \begin{cases} -K & \text{if } P^S \in [0, h_1) \\ -\frac{D_1}{a_1} & \text{if } P^S \in [h_1, h_2^1) \\ -\frac{D_1}{a_1} & \text{if } P^S \in [h_2^1, b) \\ -(\frac{D_1}{a_1} - Q) & \text{if } P^S \in [b, h_2^2) \\ -(\frac{D_2}{a_2} - Q) & \text{if } P^S \in [h_2^2, h_3) \\ 0 & \text{if } P^S \in [h_3, +\infty) \end{cases}$$

Since  $K$  is a very large number, and we have  $\frac{D_2}{a_2} \leq \frac{D_1}{a_1}$  by definition,  $\pi(Q; \mathbf{D}, P^S)$  is convex decreasing in  $P^S$ . This concludes the proof. ■

**Proof of Proposition 10:** We have  $V(Q^1, Q^2) = \mathbb{E}[\pi(Q^1, Q^2; \tilde{P}^S)] - \beta^1 Q^1 - \beta^2 Q^2$  where  $\pi(Q; P^S) = \max_{0 \leq z \leq 1'_{\mathbf{q}}} \Pi(z)$  and  $\Pi(z)$  is as given in (6). From the basic linear programming theory, it follows that  $\Pi(z)$  is concave in  $\mathbf{Q}' = (Q^1, Q^2)$ , and since concavity is preserved under maximization,  $\pi(\mathbf{Q})$  is also concave in  $\mathbf{Q}$ . Therefore,  $V(Q^1, Q^2)$  is jointly concave in  $(Q^1, Q^2)$ . The characterization of the optimality condition depends on the ordering between  $b^1 \leq b^2$  and  $h_1 \leq h_2 \leq h_3$ , thus we have ten cases to consider. We only analyze the case  $b^1 \leq h_1 \leq h_2 \leq b^2 \leq h_3$ . The other cases can be analyzed in a similar fashion.

We will only provide the sketch of the proof. Using Proposition 2, the stage 1 optimal profit  $\pi(Q; \mathbf{D})$  can be characterized for a given  $\mathbf{D}$ . Then stage 0 expected profit  $V(Q^1, Q^2)$  can be characterized for a given  $(Q^1, Q^2)$ . We obtain  $\frac{\partial V(Q^1, Q^2)}{\partial Q^1} - \frac{\partial V(Q^1, Q^2)}{\partial Q^2} =$

$$-\beta^1 - b^1 + \beta^2 + G(h_1, b^2)\chi(Q^1 \leq \frac{D_1}{a_1}) + G(h_1, h_2)\chi(\frac{D_1}{a_1} < Q^1 \leq \frac{D_2}{a_2}) + h_1\chi(Q^1 > \frac{D_2}{a_2}) \quad (21)$$

where  $\chi(\cdot)$  is the indicator function. Using (21), we define an equivalent formulation to  $V(Q^1, Q^2)$ . Let  $Q$  denote the total number of contracts to sign and  $V(Q)$  denote the expected stage 0 profit. For example, if  $\beta^1 + b^1 - \beta^2 - G(h_1, b^2) > 0$ ; it follows from (21) that  $\frac{\partial V(Q^1, Q^2)}{\partial Q^1} < \frac{\partial V(Q^1, Q^2)}{\partial Q^2}$  for all  $(Q^1, Q^2)$ . Therefore, in the optimal solution, the firm will first start using the second contract and consider switching to the first contract until after the second contract is exhausted, i.e. for  $Q > \bar{Q}^2$ . Notice that  $\beta^1 + b^1 - \beta^2 - G(h_1, b^2) = [-\beta^2 - b^2 + G(b, h_3)] - [\beta^1 - b^1 + G(h_1, h_3)]$ , i.e. it is the difference between the two contract indices given in the Proposition. Since (21) depends on  $Q^1$ , we have four cases to analyze: *i*)  $\beta^1 + b^1 > \beta^2 + G(h_1, b^2)$ , *ii*)  $\beta^2 + G(h_1, b^2) \geq \beta^1 + b^1 > \beta^2 + G(h_1, h_2)$ , *iii*)  $\beta^2 + G(h_1, h_2) \geq \beta^1 + b^1 > \beta^2 + G(h_1, h_1)$ , *iv*)  $\beta^1 + b^1 < \beta^2 + G(h_1, h_1)$ . For brevity, we only discuss the third case.

For  $\beta^2 + G(h_1, h_2) \geq \beta^1 + b^1 > \beta^2 + G(h_1, h_1)$ , it follows from (21) that we have  $\frac{\partial \mathbb{E}[V(Q^1, Q^2)]}{\partial Q^1} \geq \frac{\partial \mathbb{E}[V(Q^1, Q^2)]}{\partial Q^2}$ ,  $\forall Q^1 \leq \frac{D_2}{a_2}$ , and  $\frac{\partial \mathbb{E}[V(Q^1, Q^2)]}{\partial Q^1} < \frac{\partial \mathbb{E}[V(Q^1, Q^2)]}{\partial Q^2}$ ,  $\forall Q^1 > \frac{D_2}{a_2}$ . Since we need to consider the

ordering between  $\bar{Q}^1$  and  $\frac{D_1}{a_1} \leq \frac{D_2}{a_2}$ ; there are three subcases to analyze to characterize the optimal  $Q^*$ . We only discuss the subcase  $\bar{Q}^1 \leq \frac{D_1}{a_1}$ . In this case, we have

$$V(Q) = \begin{cases} -\beta^1 Q + \mathbb{E}[\pi(Q, 0)] & \text{if } 0 \leq Q \leq \bar{Q}^1 \\ -\beta^1 \bar{Q}^1 - \beta^2(Q - \bar{Q}^1) + \mathbb{E}[\pi(\bar{Q}^1, Q - \bar{Q}^1)] & \text{if } \bar{Q}^1 < Q \leq \bar{Q}^1 + \bar{Q}^2, \end{cases}$$

i.e. the firm starts sourcing from the first contract and switches to the second one after the capacity is exhausted. We obtain

$$\frac{\partial V(Q)}{\partial Q} = \begin{cases} -\beta^1 - b^1 + G(h_1, h_3)\chi(0 \leq Q \leq \bar{Q}^1) & \text{if } 0 \leq Q \leq \bar{Q}^1 \\ -\beta^2 - b^2 + G(b^2, h_3)\chi(\bar{Q}^1 < Q \leq \min(\frac{D_1}{a_1}, \bar{Q}^1 + \bar{Q}^2)) & \text{if } \bar{Q}^1 < Q \leq \bar{Q}^1 + \bar{Q}^2 \\ +G(b^2, b^2)\chi(\min(\frac{D_1}{a_1}, \bar{Q}^1 + \bar{Q}^2) < Q \leq \bar{Q}^1 + \bar{Q}^2) & \end{cases}$$

Therefore, the optimal  $Q^*$  is characterized by

$$Q^* = \begin{cases} 0 & \text{if } \beta^1 + b^1 > G(h_1, h_3) \\ \bar{Q}^1 & \text{if } G(h_1, h_3) \geq \beta^1 + b^1 \text{ and } \beta^2 + b^2 > G(b^2, h_3) \\ \min(\frac{D_1}{a_1}, \bar{Q}^1 + \bar{Q}^2) & \text{if } G(b^2, h_3) \geq \beta^2 + b^2 \end{cases}$$

It follows from the definition of  $Q$  that, we have

$$Q^{1*} = \begin{cases} 0 & \text{if } \beta^1 + b^1 > G(h_1, h_3) \\ \bar{Q}^1 & \text{if } G(h_1, h_3) \geq \beta^1 + b^1 \end{cases}$$

$$Q^{2*} = \begin{cases} 0 & \text{if } \beta^2 + b^2 > G(b^2, h_3) \\ \min((\frac{D_1}{a_1} - \bar{Q}^1)^+, \bar{Q}^2) & \text{if } G(b^2, h_3) \geq \beta^2 + b^2 \end{cases}$$

which is identical to the optimal  $Q^1$  and  $Q^2$  given in the Proposition for the case of  $\beta^2 + G(h_1, h_2) \geq \beta^1 + b^1 > \beta^2 + G(h_1, h_1)$  and  $\bar{Q}^1 \leq \frac{D_1}{a_1}$ .

In summary, we characterize an equivalent formulation to  $V(Q^1, Q^2)$  by focusing on the total contract volume  $Q$  from both contracts. If the marginal profit of sourcing the first unit from the first contract is higher than the marginal profit of sourcing the same from the second contract, then in the optimal solution, the firm always prefers sourcing from the first contract until its capacity is exhausted before switching to the second contract. The exact contract volume is determined by comparing the marginal profit with 0, i.e. first contract may have a larger marginal profit than the second, but it may be smaller than 0, thus, the firm does not contract, i.e.  $Q^* = 0$ . ■

**Proof of Proposition 11** : The proof is similar to Proposition 10, and is omitted. ■

## B References

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